Identify and Estimate Causal Effects of a Continuous Treatment Using Discrete Instruments

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Abstract

This paper identifies and estimates causal effects of a continuous variable, or treatment, by exploiting distributional changes in the treatment in response to a binary or discrete instrument. We explore two alternative assumptions regarding the heterogeneity of the instrument's first-stage effect on the treatment: LATE-type monotonicity and treatment rank similarity, which have been studied in distinct strands of literature. Under treatment rank similarity, we derive simple estimands for average treatment effects at different treatment quantiles, capturing treatment effect heterogeneity across treatment levels. Additionally, we propose a doubly robust causal estimand that identifies a weighted average treatment effect for all units responsive to the instrument when either of these two non-nested assumptions holds. Our doubly robust framework subsumes LATE-type estimands as a special case. We also provide asymptotically normal semiparametric estimators and demonstrate the proposed methods in an empirical application estimating the effects of sleep on well-being.

JEL codes: C14, C21, I30

Keywords: Doubly Robust Identification, Continuous treatment, Treatment distribution, LATE, Monotonicity, Rank similarity, Sleep time, Well-being

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1 Introduction

Many empirical studies aim to estimate causal effects of continuously distributed endogenous variables (or treatments), such as air pollution concentration, poverty rates, income, prices, birth weights, and time use. A common approach is to apply two-stage least squares (2SLS) using a binary or discrete instrumental variable (IV). When the IV is binary, the 2SLS estimator (without considering covariates) effectively computes the Wald ratio:

$$\tau^{Wald} := \frac{\mathbb{E}[Y|Z=1] - \mathbb{E}[Y|Z=0]}{\mathbb{E}[T|Z=1] - \mathbb{E}[T|Z=0]}, \tag{1}$$

where Y is the outcome of interest, Z is the binary instrument and T is the treatment. For τ^{Wald} to be feasible, the treatment must exhibit a non-zero mean change across IV levels. Furthermore, when treatment effects are heterogeneous and individuals select treatment intensity based on idiosyncratic gains, the causal interpretation of τ^{Wald} relies on a monotonicity assumption, i.e., the treatment changes in one direction in response to the IV changes. This monotonicity assumption is introduced in Imbens and Angrist (1994). They show that under monotonicity, τ^{Wald} identifies a local average treatment effect (LATE) for a binary treatment.

The simplicity and intuitiveness of the 2SLS approach make it widely popular. However, for continuous treatments, 2SLS may overlook where the true changes are in the treatment distribution. Policies often aim to shift one or two tails of the treatment distribution or change variance rather than the mean (e.g., minimum wage laws, pollution ceilings, or minimum capital requirements). As a result, treatment changes concentrate at specific treatment quantiles. Solely considering mean changes may lead to weak identification or even identification failure.

To address this limitation, we explore distributional changes in the first stage, drawing on insights from the non-separable IV literature, See, e.g., Chesher (2001, 2002, 2003), Imbens and Newey (2002, 2009), and Florens et al., (2008). These existing studies generally require a contintuous IV and seek to identify some structural parameters. The commonly employed first-stage restriction in the literature is treatment rank invariance or more generally treatment rank similarity.

The results have not been widely used by empirical researchers. One of the reasons might be that continuous IVs are rare in practice. We therefor focus on the empirically relevant case of binary or discrete IVs. For example, in randomized experiments - the ideal setting for the causal IV model used in this paper - researchers typically randomize discrete, most commonly binary, IVs.

We identify average effects at specific treatment quantiles, as well as weighted averages of these quantile-specific effects, under treatment rank invariance or similarity. The estimand for the former resembles the Wald ratio but is conditional on observed treatment ranks. Furthermore, we establish a doubly robust (DR) identification result: the estimand identifies aggregate causal effects for all individuals responsive to IV changes (the largest subpopulation for which causal identification is possible) under either LATE-type monotonicity or treatment rank similarity. When monotonicity holds, the DR estimand aligns with the LATE estimand. Otherwise, it identifies a weighted average treatment effect for the subpopulation affected by the IV under treatment rank similarity. Notably, the DR estimand remains valid even when the mean treatment change is zero, as long as distributional changes exist.

Both the LATE monotonicity and treatment rank similarity impose restrictions on first-stage IV effects. While these assumptions are mutually exclusive, each aligns with specific treatment selection behaviors and is refutable but not verifiable based on their testable implications (e.g., Angrist and Imbens, 1995; Dong and Shu, 2018)[1]. That is, existing tests cannot establish their validity even though they may show that these assumptions are invalid. Our DR estimand thus provides a flexible tool for causal inference under more general conditions.

Our identification is nonparametric in that we consider non-separable models for both the first-stage and the outcome equation. Non-separable models allow for treatment effect heterogeneity and individuals self-selection of different treatment levels based on idiosyncratic gains, both of which are important features of the data as supported by economic theory and empirical evidence. Based on our identification results, we propose simple estimators for average effects at different

[1]For the testable implication of the LATE-type monotonicity when treatment is multi-valued, see, e.g., Angrist and Imbens (1995) and Fiorini and Stevens (2021). For tests of the testable implication of rank similarity, see Dong and Shu (2018) and Frandsen and Lefgren (2018).

treatment quantiles (conditional or unconditional on covariates) and a DR estimator for weighted average effects. These estimators are shown to be consistent and asymptotically normal. We illustrate the utility of our methods by analyzing the impact of sleep duration on well-being using data from Bessone et al. (2021). In this empirical application, we uncover nuanced insights into treatment effect heterogeneity across different treatment levels and demonstrate how the DR approach complements traditional IV or 2SLS estimates.[2]

This paper builds upon and integrates two distinct strands of literature: the LATE framework and the non-separable IV model. The LATE model is introduced in Imbens and Angrist (1994), which is further extended in Angrist and Imbens (1995), Angrist, Imbens and Rubin (1996), Angrist et al. (2000), Abadie (2003), Frölich (2007), de Chaisemartin (2017), Dahl, Huber, and Mellace (2023), etc. The LATE model relies on the monotonicity assumption mentioned previously or some weaker versions of it for causal identification. Many studies in the non-separable IV model literature explore rank invariance or rank similarity in the first stage for causal identification. In addition to the aforementioned Chesher (2001, 2002, 2003), Imbens and Newey (2002, 2009), and Florens et al. (2008), more recently Torgovitsky (2015), and D'Haultfoeuille and Février (2015) provide detailed discussions of the identifying power of rank restrictions in the treatment and/or in the outcome equation. In addition, Masten and Torgovitsky (2016) consider a random correlated coefficients model and utilize treatment rank invariance to identify the average partial effect of continuous treatment variables, using binary or discrete instruments. For the DR identification approach, a few existing studies take this approach, see, e.g., Dong, Lee, and Gou (2023) and Arkhangelsky and Imbens (2022). Both papers are set in different frameworks than the current one. Dong, Lee, and Gou (2023) study the regression discontinuity design, while Arkhangelsky and Imbens (2022) investigate the panel data model.[3]

The rest of the paper proceeds as follows. Section 2 presents identification results for the basic

[2]The replication package, including Stata code for our estimators and a link to the data, is available upon request.

[3]The current paper differs from the regression discontinuity setup of Dong, Lee, and Gou (2023) in multiple dimensions, including allowing the IV independence and treatment rank similarity to hold conditional on a vector of continuous and/or discrete covariates, allowing for a multi-valued IV or a vector of discrete IVs and completely different estimation and inference procedures.

setup without covariates. Section 3 extends these results to accomodate covariates. Section 4 develops semiparametric estimators and establishes their consistency and asymptotic normality. Section 5 discusses extensions to a multi-valued IV or a vector of discrete IVs, with or without covariates. Section 6 presents our empirical application. Section 7 concludes.

2 Identification in the Basic Setup

Let $Y \in \mathcal{Y} \subset \mathcal{R}$ be the outcome of interest, e.g., a measure of well-being. Y can be continuous or discrete. Let $T \in \mathcal{T} \subset \mathcal{R}$ be a continuous treatment variable, e.g., sleep time. Let $Z \in \{0, 1\}$ be a binary IV for T, e.g., an indicator for being randomly assigned to a group receiving encouragement or financial incentives to increase night sleep.

To present the core ideas, we suppress all the covariates in this section. The general setup with covariates is presented in the next section. Assume that Y and T are generated as

$$
\begin{aligned}
Y &= g(T, \varepsilon), &(2)\\
T &= h(Z, U), &(3)
\end{aligned}
$$

where ε captures all the other factors other than T that affect Y, and similarly U captures all the other reduced-form factors other than Z that affect T. The outcome disturbance $\varepsilon \in \mathcal{E} \subset \mathcal{R}^{d_\varepsilon}$ is allowed to be of arbitrary dimension, so d_ε does not need to be finite. Without loss of generality, rewrite eq. (3) as

$$
T = ZT_1(U_1) + (1 - Z)T_0(U_0), \tag{4}
$$

where $T_z(\cdot), z = 0, 1$ are some unknown functions, and the reduced-form disturbance $U_z \in \mathcal{U}_z \subset \mathcal{R}$, $z = 0, 1$. Later we impose an assumption that essentially requires $T_z(\cdot)$ to be the quantile functions and $\mathcal{U}_z$ to be the rank variables. Note by construction $U = ZU_1 + (1 - Z)U_0$.

Define $Y_t := g(t, \varepsilon)$ as the potential outcome when T is exogenously set to be t. Further define $T_z := T_z(U_z)$, $z = 0, 1$, as the potential treatment when Z is exogenously set to be z. Denote the support of T_z as $\mathcal{T}_z$. The observed treatment is then $T = ZT_1 + (1 - Z)\, T_0$. We use $F_\cdot(\cdot)$ and $F_{\cdot|\cdot}(\cdot|\cdot)$ to denote the unconditional cumulative distribution function (CDF) and conditional CDF, respectively.

Assumption 1 (Treatment quantile representation)**.** *$T_z(u)$ is strictly increasing in u, and $U_z \sim Unif\,(0,1)$, $z = 0, 1$.*

Assumption 1 requires that the potential treatment T_z is continuous with a strictly increasing CDF. The condition $U_z \sim Unif\,(0,1)$ involves a normalization. This kind of normalization is necessary, since the identification results hold up to a monotonic transformation of U_z, as long as U_z is continuous with a strictly increasing CDF. See discussions in Matzkin (2003) and more recently Torgovitsky (2015). By Assumption 1, $T_z(u)$ is the u quantile of T_z, and $U_z = F_{T_z}(T_z)$ is the rank of the potential treatment. Further, $U = ZU_1 + (1 - Z)\, U_0$ is the observed treatment rank.

Assumption 2 (Independence)**.** $Z \perp (U_z, \varepsilon)$, $z = 0, 1$.

Assumption 2 essentially requires Z to be randomly assigned. More generally, we can allow the independence condition to hold only after conditioning on relevant pre-determined covariates, which we will discuss in the next session. Assumptions 1 and 2 imply $U \perp Z$, because for $z = 0, 1$, $\Pr(U \leq \tau | Z = z) = \Pr(U_z \leq \tau | Z = z) = \Pr(U_z \leq \tau) = \tau$, where the last equality follows the condition $Z \perp U_z$ as implied by Assumption 2.

Assumption 3 (Monotonicity)**.** $\Pr(T_1 \geq T_0) = 1$.

Assumption 3 requires that treatment can only change in one direction when Z changes - without loss of generality, we normalize it to be non-decreasing. For example, this assumption holds in the usual linear regression model of T with a constant coefficient on Z.

Assumption 3 can not be tested directly, but it has testable implications. It implies $T_1(u) - T_0(u) \geq 0$ for all $u \in (0, 1)$, i.e., T_1 stochastically dominates T_0. Since stochastic dominance is a necessary but not sufficient condition for Assumption 3, rejecting stochastic dominance could mean monotonicity does not hold, but failing to reject does not necessarily mean that monotonicity holds. That is, Assumption 3 is refutable but not verifiable.

Assumption 4 (First-stage)**.** *The set* $\{u \in (0,1) : T_1(u) \neq T_0(u)\}$ *has positive Lebesgue measure.*

Assumption 4 requires that the distribution of T changes with Z. Assumption 4 is strictly weaker than the standard first-stage assumption of the LATE model, which requires $\mathbb{E}[T_1] \neq \mathbb{E}[T_0]$. For example, when the policy instrument Z affects the variance or shifts the tails of the treatment distribution but otherwise leaves the average treatment level unaffected, we have the standard LATE first-stage assumption fails, but the above Assumption 4 holds.

$\Pr(T_1 \neq T_0) = 1$, which is further equivalent to $\Pr(T_1 = T_0) = 0$

Assumptions 2, 3 and 4 together imply $\mathbb{E}[T|Z=1] - \mathbb{E}[T|Z=0] > 0$. For convenience of exposition, we generalize the standard definition of compliers, which is defined for a binary treatment (Angrist, Imbens, and Rubin, 1996). Let $\mathcal{T}_c = \{(t_0, t_1) \in \mathcal{T}_0 \times \mathcal{T}_1 : t_1 - t_0 > 0\}$ be the set of all types of compliers. Define $LATE^c(t_0, t_1) = \mathbb{E}\left[\frac{Y_{t_1} - Y_{t_0}}{t_1 - t_0} | T_1 = t_1, T_0 = t_0\right]$ for any $(t_0, t_1) \in \mathcal{T}_c$. $LATE^c(t_0, t_1)$ is the local average treatment effect for complier type $(t_0, t_1) \in \mathcal{T}_c$. For example, in the case of a binary treatment, $\tau^{Wald} = LATE^c(0,1)$. More generally when treatment is continuous as in our setup, τ^{Wald} is a weighted average of $LATE^c(t_0, t_1)$ for all $(t_0, t_1) \in \mathcal{T}_c$. We formalize this result in the following lemma.

Lemma 1. *If Assumptions 1 - 4 hold, then*

$$\tau^{Wald} = \iint_{\mathcal{T}_c} w_{t_0,t_1} LATE^c(t_0, t_1) F_{T_0,T_1}(dt_0, dt_1)$$

where $w_{t_0,t_1} = (t_1 - t_0) / \iint_{\mathcal{T}_c} (t_1 - t_0) F_{T_0,T_1}(dt_0, dt_1)$.

The above lemma states that under Assumptions *1-4*, τ^{Wald} in eq. (1) identifies a weighted average of the average treatment effects for different compliers, where the weights are proportional to their treatment intensity change $(t_1 - t_0)$. Frölich (2007) gives a comparable expression when treatment is multi-valued.

When $g(T, \varepsilon)$ is continuously differentiable in its first argument, the identified causal parameter can be further expressed as a weighted average derivative of Y w.r.t. T, following Angrist et al. (2000, Theorem 1). The exact form of the weighted average derivative is provided in the proof of Lemma 1 in the Appendix.[4]

In the following, we provide an alternative assumption to Assumption 3, which allows us to identify treatment effect heterogeneity at different treatment levels.

Assumption 5 (Treatment Rank Similarity)**.** $U_0|\varepsilon \sim U_1|\varepsilon$.

Assumption 5 states that if two individuals with the same ε value, then the probability distribution of their potential treatment ranks stays the same. Without conditioning on ε, U_0 and U_1 both follow a uniform distribution over the unit interval due to normalization, so $F_{U_0}(u) = F_{U_1}(u)$ by construction. Assumption 5 implies $\varepsilon|U_0 = u \sim \varepsilon|U_1 = u$ by Bayes' theorem, so ε has the same distribution at the same rank of the potential treatment.

A slightly stronger assumption is treatment rank invariance, i.e., $U_0 = U_1$. Treatment rank invariance holds trivially when the treatment model is additively separable in a scalar disturbance, but this assumption does not require additive separability in general. Treatment rank invariance sometime is stated as monotonicity in a scalar disturbance in the treatment model (see, e.g., Imbens and Newey, 2009). Treatment rank similarity in Assumption 5 relaxes treatment rank invariance - instead of assuming the ranks of the potential treatments to be the same, it only assumes that they have the same conditional probability distribution for any given ε, and thereby permits random deviations from the common rank level between the potential treatments. For example, if the common rank level for night sleep (the actual time one is in sleep as measured by actigraphy) is determined by individuals' biological clock (possibly after conditioning on observable covariates as discussed in our general setup), which does not change with Z, then rank similarity permits that the increase in night sleep is subject to some random factors.

Lemma 2. *Under Assumptions 1, 2 and 5,* $T \perp \varepsilon | U$.

[4] Our weighted average derivative differs from that of Angrist et al. (2000), who define theirs in terms of overlapping subpopulations. However, the two expressions are ultimately equivalent.

Lemma 2 suggests that U is a control variable as defined by Imbens and Newey (2009), i.e., conditional on the observed treatment rank U, T is exogenous to Y. Intuitively, under Assumptions 2 and 5 and holding U fixed, the only variation in T is the exogenous variation induced by Z.[5]

Based on Lemma 2, one may condition on U in the outcome equation to estimate the causal effect of T on Y. Let $\mathcal{U} = \{u \in (0,1) : T_1(u) \neq T_0(u)\}$. For any $u \in \mathcal{U}$, define

$$\tau(u) := \frac{\Delta Y(u)}{\Delta T(u)}. \tag{5}$$

where for $M = Y, T$,

$$\Delta M(u) = \mathbb{E}[M|Z=1, U=u] - \mathbb{E}[M|Z=0, U=u].$$

The numerator captures the reduced-form effect of Z on Y given $U = u$, while the denominator captures the first-stage treatment change given $U = u$. The corresponding estimator (by replacing the population means and ranks by their sample analogues) is analogous to the indirect least square estimator in the linear IV model setting.

Note that conditional on $U = u$, with a binary Z, the treatment T can potentially take two values $T_0(u)$ and $T_1(u)$. When T changes exogenously from $T_0(u)$ and $T_1(u)$, the corresponding average effect on the outcome is $\mathbb{E}\left[Y_{T_1(u)} - Y_{T_0(u)}|U = u\right]$. The following theorem clarifies what $\tau(u)$ identifies.

Theorem 1. *If Assumptions 1, 2, 4 and 5 hold, then for any $u \in \mathcal{U}$,*

$$\begin{aligned} \tau(u) &= \mathbb{E}\left[\frac{Y_{T_1(u)} - Y_{T_0(u)}}{T_1(u) - T_0(u)}|U = u\right] && (6) \\ &= \int \{g(T_1(u), e) - g(T_0(u), e)\} \frac{1}{T_1(u) - T_0(u)} F_{\varepsilon|U}(de|u). && (7) \end{aligned}$$

[5] This result is closely related to Theorem 1 of Imbens and Newey (2009), except that we assume rank similarity instead of rank invariance and that we focus on a binary IV instead of an IV that may have a large support. The large support is required to identify structural parameters, like the average structural function, when the outcome disturbance is of arbitrary dimension.

Theorem 1 shows that $\tau(u)$ captures an average (per unit) treatment effect at the u quantile of the treatment, so it can be used to measure treatment effect heterogeneity at different treatment intensities. The denominator in eq. (6) reflects the fact that $T_z(u) \notin \{0, 1\}$ in general. In the integral in eq. (7), T exogenously changes from $T_0(u)$ to $T_1(u)$ while holding ε fixed at e, so $\tau(u)$ is causal from a *ceteris paribus* point of view.

To see the results in Theorem 1, note

$$\begin{aligned}
\mathbb{E}[Y|Z=1, U=u] &= \mathbb{E}\left[g\left(T_1(u), \varepsilon\right)|Z=1, U=u\right] \\
&= \mathbb{E}\left[g\left(T_1(u), \varepsilon\right)|U=u\right] \\
&= \mathbb{E}\left[Y_{T_1(u)}|U=u\right]
\end{aligned}$$

where the first equality follows from the models of Y and T, (2) and (4), respectively, the second equality follows from the condition $Z \perp \varepsilon|U$ as shown in the proof of Lemma 2, and the last equality is by the definition of the potential outcome. One can similarly show $\mathbb{E}[Y|Z=0, U=u] = \mathbb{E}\left[Y_{T_0(u)}|U=u\right]$. That is, we can identify $\mathbb{E}\left[Y_{T_z(u)}|U=u\right]$ for $z = 0, 1$ and $u \in \mathcal{U}$. Ideally one may wish to recover $\mathbb{E}[Y_t]$ for any $t \in \mathcal{T}$, which is known as the average dose-response or structural function. However, it is impossible to identify $\mathbb{E}[Y_t]$ for any $t \in \mathcal{T}$ without further assumptions, since we have a binary instrument and we do not restrict the dimensionality of the outcome disturbance, i.e., we do not impose rank invariance in the outcome model.

Let $q_z(u) = F_{T|Z}^{-1}(u|z)$ be the conditional u quantile of T given $Z = z$, and further $\Delta q(u) = q_1(u) - q_0(u)$. By eq. (4) and Assumptions 1 and 2, U and T follow a one-to-one mapping given $Z = z$, and conditioning on $U = u$ is the same as conditioning on $T = q_z(u)$. Let the conditional mean function of Y given Z and T be $m_z(t) = \mathbb{E}[Y|Z=z, T=t]$, $z = 0, 1$. One can alternatively write $\tau(u)$ in eq. (5) as

$$\tau(u) = \frac{m_1(q_1(u)) - m_0(q_0(u))}{q_1(u) - q_0(u)}. \tag{8}$$

We use eq. (8) to construct our estimator later.

Oftentimes, researchers or policy makers are interested in some summary measure of the overall treatment effect. With $\tau(u)$, one can further identify and estimate a weighted average of $\tau(u)$, i.e.,

$$\tau^{RS}(w) := \int_{\mathcal{U}} \tau(u)w(u)du$$

for any known or estimable weighting function $w(u)$ such that $w(u) \geq 0$ and $\int_{\mathcal{U}} w(u)du = 1$. The weighting function $w(u)$ must be non-negative; otherwise, $\tau^{RS}(w)$ may represent a weighted difference rather than a weighted average of the treatment effects for different units. For example, if $\mathcal{U} = (0, 1)$, and one chooses $w(u) = 1$, then $\tau^{RS}(w) = \mathbb{E}[\tau(U)]$.

Under treatment rank similarity (Assumption 5), $\tau^{RS}(w)$ represents a weighted average of the treatment effects for all units responsive to the instrument. By Lemma 1, τ^{Wald} is a weighted average of the treatment effects for all units responsive to the instrument under monotonicity (Assumption 3). Notably, both assumptions impose restrictions on the heterogeneity of the first-stage IV effect: monotonicity enforces a sign restriction, whereas treatment rank similarity imposes a rank restriction. Neither assumption implies the other. Moreover, while these assumptions may be subject to empirical refutation, they cannot be directly verified. To address this, we consider a weighting function that ensures a doubly robust (DR) property for the resulting estimand, meaning the estimand remains valid under either of the two alternative identifying assumptions.

Proposition 1. *Let Assumptions 1, 2 and 4 hold. Furthermore, if either Assumption 3 or Assumption 5 holds, then* $\tau^{DR} := \int_{\mathcal{U}} \tau(u)w^{DR}(u)du$ *for* $w^{DR}(u) = |\Delta q(u)| / \int_{\mathcal{U}} |\Delta q(u)|du$ *identifies a weighted average of the average treatment effects among all the units for which* $T_1 \neq T_0$.

Proposition 1 synthesizes the results of Lemma 1 and Theorem 1. It shows that under either first-stage restriction on the IV effect heterogeneity, τ^{DR} identifies a weighted average of the treatment effects for all units that respond to the IV change. These units constitute the largest identifiable subpopulation for which causal effects can be determined without imposing additional restrictions. The two alternative first-stage assumptions define the nature of these responses: either units change their treatment in a monotonic manner, or their treatment ranks retain the same probability distribution.

When monotonicity (Assumption 3) holds, $w^{DR}(u) = \Delta q\,(u) \,/\, \int_{\mathcal{U}} \Delta q\,(u)\,du$. Consequently,

$$\begin{aligned} \tau^{DR} &= \frac{\int_0^1 \{\mathbb{E}[Y|Z=1,U=u] - \mathbb{E}[Y|Z=0,U=u]\}\,du}{\int_0^1 \Delta q\,(u)\,du} \\ &= \frac{\mathbb{E}[Y|Z=1] - \mathbb{E}[Y|Z=0]}{\mathbb{E}[T|Z=1] - \mathbb{E}[T|Z=0]} \\ &= \tau^{Wald} \end{aligned}$$

Thus, τ^{DR} reduces to the standard LATE estimand τ^{Wald}, as given by eq. (1), when monotonicity holds. By Lemma 1, in this case, τ^{DR} identifies a weighted average of the average treatment effects for different compliers. On the other hand, when monotonicity (Assumption 3) does not hold, but rank similarity (Assumption 5) does, τ^{DR} becomes a weighted average of $\tau(u)$ for $u \in \mathcal{U}$, and by Theorem 1, $\tau(u)$ captures the average treatment effect at the u quantile of treatment. In either case, τ^{DR} identifies a weighted average of the average treatment effects for all units that adjust their treatment levels in response to changes in the IV. The weights are proportional to the magnitude of their treatment changes.

The weighting function in Proposition 1 allows $\Delta q\,(u)$ to change signs, indicating that the LATE monotonicity condition does not hold. Consequently, τ^{DR} may average over two distinct groups: those who increase their treatment levels and those who decrease them in response to changes in the IV[6] Since individual treatment response types are not point-identified, it is not possible to separately identify causal effects for each type. However, if desired, one can define DR estimands separately for treatment quantiles where $\Delta q\,(u) > 0$ and those where $\Delta q\,(u) < 0$. We explore this further in Section S.2 in the Appendix.

So far, we have focused our discussion on (weighted) average effects. With a continuous outcome, one may easily extend the above identification results to identify $F_{Y_{T_z(u)|U}}(y|u)$, $z = 0, 1$ and hence the average distributional effects at a given treatment quantile. In particular, for any

[6] This issue is not unique to our setting; it arises whenever a researcher estimates average effects while allowing treatment changes to switch signs.

$u \in \mathcal{U}$, $F_{Y_{T_z(u)}|U}(y|u) = \mathbb{E}\left[1(Y \leq y)|U = u, Z = z\right]$, $z = 0, 1$. One may further develop an analogous DR estimand for the effect of T on $1(Y \leq y)$ for any $y \in \mathcal{Y}$.

3 Identification with Covariates

The previous section introduces our core idea without accounting for covariates. However, there are at least three compelling reasons to incorporate covariates into the analysis. First, conditioning on covariates is necessary when the IV itself is confounded, such that the IV assumptions are valid only conditional on covariates. Second, rank similarity is more plausible when conditioned on all relevant covariates, allowing the remaining model error to be treated as a scalar. For further discussion on this point, see Chernozhukov and Hansen (2005, 2006). Third, the first-stage monotonicity assumption can be relaxed by permitting the direction of monotonicity to vary with covariates.

When covariates are included in the non-separable models for Y and T, i.e., (2) and (3), and all previous assumptions hold conditional on covariates, the earlier results naturally extend to this conditional framework. However, these conditional results may have limited practical utility, as presenting all conditional findings can become cumbersome, especially when dealing with many continuous covariates. In this section, we aim to identify unconditional weighted average effects while accommodating the presence of covariates.

Let $X \in \mathcal{X} \subset \mathcal{R}^{d_X}$ denote the vector of covariates. We consider the following models for Y and T:

$$\begin{aligned} Y &= G(T, X, \epsilon), && (9) \\ T &= H(Z, X, V) \\ &= ZT_1(X, V_1) + (1 - Z)\, T_0(X, V_0), && (10) \end{aligned}$$

where by construction $V = V_1 Z + V_0(1 - Z)$.

As before, $T_z := T_z(X, V_z)$ is the potential treatment when Z is exogenously set to be $z \in \{0, 1\}$ and $Y_t := G(t, X, \epsilon)$ is the potential outcome when T is exogenously set to be $t \in \mathcal{T} \subset \mathcal{R}$. We extend Assumptions 1 - 5 to condition on covariates X as follows.

Assumption C1 (Conditional Treatment Quantile)**.** *For any* $x \in \mathcal{X}$*,* $T_z(x, v)$*,* $z = 0, 1$*, is strictly increasing in* v*, and* $V_z \sim Unif\,(0, 1)$*.*

By Assumption C1, $T_z\,(x, v)$ is the conditional quantile function of T_z given X, and $V_z = F_{T_z|X}\,(T_z|X)$ is the conditional rank of T_z given X.

Assumption C2 (Conditional Independence)**.** $Z \perp (V_z, \epsilon)\,|X, z = 0, 1$.

Assumption C3 (Conditional Monotonicity)**.** *Either* $\Pr\,(T_1 \geq T_0|X = x) = 1$ *or* $\Pr\,(T_1 \leq T_0|X = x) = 1$ *for any* $x \in \mathcal{X}$*.*

Assumption C4 (Conditional First-stage)**.** *For at least some* $x \in \mathcal{X}$*, the set* $\{v \in (0, 1) : T_0\,(x, v) \neq T_1\,(x, v)\}$ *has positive Lebesgue measure.*

Assumption C5 (Conditional Treatment Rank Similarity)**.** $V_1|(\epsilon, X) \sim V_0|(\epsilon, X)$.

Assumption C6 (Common Support)**.** $\Pr\,(Z = 1|X = x) \in (0, 1)$ *for any* $x \in \mathcal{X}$*.*

Assumption C2 requires Z to be unconfounded, instead of being randomly assigned as required by Assumption 2. Assumption C2 requires conditional independence, instead of the stronger joint or full independence $(V_z, \epsilon, X) \perp Z$, so exogeneity of X is not required - e.g., X can be correlated with Z. Assumption C3 allows the direction of monotonicity to change with covariates, which relaxes its unconditional counterpart Assumption 3. Note that under Assumptions C1-C4 and C6, if Assumption C3 holds, then the direction of the monotonicity can be identified by the sign of $\mathbb{E}\,[T|Z = 1, X = x] - \mathbb{E}\,[T|Z = 0, X = x]$. Assumption C6 is a common support assumption to ensure that all the parameters we consider are well-defined. In addition, Assumption C5 requires that treatment rank similarity holds only among the subgroup of units with the same observed covariate values, which is weaker than Assumption 5.[7] The following Lemma extends Lemma 2 to allow for covariates.

[7]Note that V_z is defined conditionally on X, while U_z is defined unconditionally. Given that X are determinants of Y, one can let X be an observable sub-vector of ε in $Y = g\,(T, \varepsilon)$. That is, $\varepsilon = (X, \epsilon)$. Assumption 5 $U_1|\varepsilon \sim U_0|\varepsilon$ implies $U_1|X \sim U_0|X$, so $F_{U_1|X}\,(u|x) = F_{U_0|X}\,(u|x)$ for any $u \in (0, 1)$ and $x \in \mathcal{X}$. It fol-

Lemma 3. *Under Assumptions C1, C2 and C5, $T \perp \epsilon | (V, X)$.*

Lemma 3 is a conditional (on X) version of Lemma 2. It establishes that V is a control variable given X. For $M = Y, T$, let

$$\Delta M(x, v) = \mathbb{E}[M|Z = 1, X = x, V = v] - \mathbb{E}[M|Z = 0, X = x, V = v].$$

The resulting IV estimand, conditional on $X = x$ and $V = v$, is given by

$$\pi(x, v) := \frac{\Delta Y(x, v)}{\Delta T(x, v)}, \tag{11}$$

whenever $\Delta T(x, v) \neq 0$.

Let $q_z(x, v) = F^{-1}_{T|Z,X}(v|z, x)$ be the conditional v quantile of T given $Z = z$ and $X = x$. By eq. (10) and Assumptions C1 and C2, T and V follow a one-to-one mapping given $Z = z$ and $X = x$, i.e., conditioning on $V = v$ is the same as conditioning on $T = q_z(x, v)$ in (11). Then $\pi(x, v)$ can be re-written as

$$\pi(x, v) = \frac{m_1(x, q_1(x, v)) - m_0(x, q_0(x, v))}{q_1(x, v) - q_0(x, v)},$$

where $m_z(x, t) = \mathbb{E}[Y|Z = z, X = x, T = t]$.

Let $\Delta q(x, v) = q_1(v, x) - q_0(v, x)$. By construction, $\Delta T(x, v) = \Delta q(x, v)$. Assumptions C1 and C6 ensure that $\Delta q(x, v)$ is well defined for all $x \in \mathcal{X}$ and $v \in (0, 1)$. Let $\mathcal{S} = \{(x, v) \in \mathcal{X} \times (0, 1) \colon \Delta q(x, v) \neq 0\}$. We have the following Theorem 2, which extends Theorem 1.

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lows that $F_{V_0|X,\epsilon}(v|X = x, \epsilon = e) = \mathbb{E}[1(V_0 \leq v)|X = x, \epsilon = e] = \mathbb{E}\left[1\left(F_{U_0|X}(U_0|x) \leq v\right)|X = x, \epsilon = e\right] = \mathbb{E}\left[1\left(F_{U_1|X}(U_1|x) \leq v\right)|X = x, \epsilon = e\right] = F_{V_1|X,\epsilon}(v|X = x, \epsilon = e)$ for any v, x, and e in their support, where the second equality follows from $V_z = F_{Tz|X}(T_z|X)$ by Assumption C1, which can be further written as $V_z = F_{Uz|X}(U_z|X)$, $z = 0, 1$, since T_z and U_z follow a one-to-one mapping by Assumption 1. Therefore, $V_0|X, \epsilon \sim V_1|X, \epsilon$.

Theorem 2. *If Assumptions C1, C2, C4, C5 and C6 hold, then for any* $(x, v) \in \mathcal{S}$,

$$\pi(x, v) = \mathbb{E}\left[\frac{Y_{T_1(x,v)} - Y_{T_0(x,v)}}{T_1(x, v) - T_0(x, v)} | X = x, V = v\right] \tag{12}$$

$$= \int \{G(T_1(x, v), x, e) - G(T_0(x, v), x, e)\} \frac{F_{\epsilon|X,V}(de|x, v)}{T_1(x, v) - T_0(x, v)}. \tag{13}$$

Theorem 2 shows that $\pi(x, v)$ identifies a conditional weighted average treatment effect at the conditional v quantile of the treatment given $X = x$. It is clear from eq. (13) that $\pi(x, v)$ represents the causal effect of an exogenous change in treatment from $T_0(x, v)$ to $T_1(x, v)$, while holding X and ϵ fixed at x and e.

If desired, one may average $\pi(x, v)$ over the distribution of X to obtain a weighted average effect at the conditional v quantile of the treatment. For notational convenience, in the following, we assume $\pi(x, v) = 0$ when $\Delta q(x, v) = 0$, so that $\pi(x, v)$ is defined for all $(x, v) \in \mathcal{X} \times (0, 1)$. For example, for any $v \in (0, 1)$ such that $\Pr(\Delta q(X, v) \neq 0) > 0$, one can define

$$\pi(v) := \int_{\mathcal{X}} \pi(x, v) w_v(x) dx, \tag{14}$$

where $w_v(x) = |\Delta q(x, v)| f_X(x) / \int_{\mathcal{X}} |\Delta q(x, v)| f_X(x) dx$. $\pi(v)$ identifies a weighted average effect at the conditional v quantile of the treatment. In contrast, $\tau(u)$ identifies an average effect at the unconditional u quantile of the treatment. $\pi(v)$ can be useful in investigating treatment effect heterogeneity at the conditional v quantile of the treatment.

Consider now constructing a DR estimand for the overall unconditional weighted average effect based on $\pi(x, v)$. Since Z is valid only after conditioning on pre-determined covariates, τ^{Wald} is no longer a valid causal estimand. Let $\Delta M(x) = \mathbb{E}[M|Z = 1, X = x] - \mathbb{E}[M|Z = 0, X = x]$, $M = Y, T$. Further let $\mathbb{I}^+(x) = 1(\Delta T(x) \geq 0)$ and $\mathbb{I}^-(x) = 1(\Delta T(x) \leq 0)$. Define

$$\tau^{Wald\_X} := \frac{\int_{\mathcal{X}} \left\{\mathbb{I}^+(x) \Delta Y(x) - \mathbb{I}^-(x) \Delta Y(x)\right\} f_X(x) dx}{\int_{\mathcal{X}} \{\mathbb{I}^+(x) \Delta T(x) - \mathbb{I}^-(x) \Delta T(x)\} f_X(x) dx}. \tag{15}$$

A special case of Eq. (15) nder the stronger assumption of unconditional monotonicity (Assumption 3) is proposed in Frölich (2007) for binary or discrete treatments. Importantly, in eq. (15), the numerator does not simplify to $\mathbb{E}[Y|Z=1]-\mathbb{E}[Y|Z=0]$, nor does the denominator simplify to $\mathbb{E}[T|Z=1]-\mathbb{E}[T|Z=0]$, as X is not assumed to be independent of Z. Consequently, the distributions of X conditional on $Z=0$ and $Z=1$ generally differ.

Let $\mathcal{T}_{c,d}=\{(t_0,t_1)\in\mathcal{T}_0\times\mathcal{T}_1 : t_1\neq t_0\}$ be the set of all types of compliers and defiers. Note that $LATE^c(t_0,t_1)=\mathbb{E}\Big[\frac{Y_{t_1}-Y_{t_0}}{t_1-t_0}|T_1=t_1,T_0=t_0\Big]$ as written can be used for a complier average treatment effect or a defier average treatment effect, depending on the ordering of t_0 and t_1. For notational clarity, however, in this section, we relabel it and let $LATE^{c,d}(t_0,t_1)=\mathbb{E}\Big[\frac{Y_{t_1}-Y_{t_0}}{t_1-t_0}|T_1=t_1,T_0=t_0\Big]$ for $(t_0,t_1)\in\mathcal{T}_{c,d}$.

Lemma 4. *If Assumptions C1 - C4 and Assumption C6 hold, then*

$$\tau^{Wald\_X}=\iint_{\mathcal{T}_{c,d}} w_{t_0,t_1} LATE^{c,d}(t_0,t_1)\, F_{T_0,T_1}(dt_0,dt_1)$$

where $w_{t_0,t_1}=|t_1-t_0|/\iint_{\mathcal{T}_{c,d}}|t_1-t_0|F_{T_0,T_1}(dt_0,dt_1)$. *If further Assumption 3 holds,*

$$\tau^{Wald\_X}=\iint_{\mathcal{T}_c} w_{t_0,t_1} LATE^{c}(t_0,t_1)\, F_{T_0,T_1}(dt_0,dt_1),$$

where $w_{t_0,t_1}=(t_1-t_0)/\iint_{\mathcal{T}_c}(t_1-t_0)\,F_{T_0,T_1}(dt_0,dt_1)$.

Lemma 4 shows that under unconditional monotonicity (Assumption 3, which rules out defiers), τ^{Wald_X} identifies the same unconditional effect as τ^{Wald} would if Z were valid without conditioning on covariates. More generally, when monotonicity varies with covariates (Assumption C3), τ^{Wald_X} identifies a weighted average of the average effects for both compliers and defiers. Frölich (2007) establishes a similar result for binary or discrete treatments under unconditional monotonicity in the first stage. Based on Lemma 4, we construct a DR estimand incorporating τ^{Wald_X} (instead of the invlide τ^{Wald}) as a special case.

Proposition 2. *Let Assumptions C1, C2, C4 and C6 hold. Furthermore, if either C3 or C5 holds,*

$$\pi^{DR} := \iint_{\mathcal{S}} \pi(x,v) w(x,v)\, dv dx$$

for $w(x,v) = |\Delta q(x,v)| f_X(x) / \iint_{\mathcal{S}} |\Delta q(x,v)| f_X(x) dv dx$ *identifies a weighted average of the average treatment effects among all the units for which* $T_1 \neq T_0$.

Note that under conditional monotonicity, for any $(x,v) \in \mathcal{S}$, $\mathbb{I}^+(x) = 1$ implies $\Delta q(x,v) > 0$ and $\mathbb{I}^-(x) = 1$ implies $\Delta q(x,v) < 0$, so one has $|\Delta q(x,v)| = \mathbb{I}^+(x)\, \Delta q(x,v) - \mathbb{I}^-(x)\, \Delta q(x,v)$, while

$$\begin{aligned}
\pi(x,v) &= \frac{\Delta Y(x,v)}{\Delta q(x,v)} \\
&= \frac{\mathbb{I}^+(x)\, \Delta Y(x,v) - \mathbb{I}^-(x)\, \Delta Y(x,v)}{\mathbb{I}^+(x)\, \Delta q(x,v) - \mathbb{I}^-(x)\, \Delta q(x,v)},
\end{aligned}$$

so that

$$\begin{aligned}
\pi^{DR} &= \frac{\iint_{\mathcal{S}} \left\{\mathbb{I}^+(x)\, \Delta Y(x,v) - \mathbb{I}^-(x)\, \Delta Y(x,v)\right\} f_X(x) dv dx}{\iint_{\mathcal{S}} \left\{\mathbb{I}^+(x)\, \Delta q(x,v) - \mathbb{I}^-(x)\, \Delta q(x,v)\right\} f_X(x) dv dx} \\
&= \frac{\iint_{\mathcal{X}\times(0,1)} \left\{\mathbb{I}^+(x)\, \Delta Y(x,v) - \mathbb{I}^-(x)\, \Delta Y(x,v)\right\} f_X(x) dv dx}{\iint_{\mathcal{X}\times(0,1)} \left\{\mathbb{I}^+(x)\, \Delta q(x,v) - \mathbb{I}^-(x)\, \Delta q(x,v)\right\} f_X(x) dv dx} \\
&= \frac{\int_{\mathcal{X}} \left\{\mathbb{I}^+(x)\, \Delta Y(x) - \mathbb{I}^-(x)\, \Delta Y(x)\right\} f_X(x)\, dx}{\int_{\mathcal{X}} \left\{\mathbb{I}^+(x)\, \Delta T(x) - \mathbb{I}^-(x)\, \Delta T(x)\right\} f_X(x)\, dx} \\
&= \tau^{Wald_X},
\end{aligned}$$

where the second equality follows from Assumptions C3 and C4, the third equality follows from Assumption C2, which implies $V_z \perp Z | X$.

When conditional rank similarity (Assumption C5) holds, π^{DR} is a weighted average of $\pi(x,v)$ for $(x,v) \in \mathcal{S}$, which is a causal estimand by Theorem 2; Otherwise, when monotonicity (Assumption C3_) holds, $\pi^{DR} = \tau^{Wald\_X}$, which, as shown in Lemma 4, identifies a weighted average of $LATE^{c,d}(t_0, t_1)$ for $(t_0, t_1) \in \mathcal{T}_{c,d}$. Either way, π^{DR} identifies a weighted average of the average treatment effects for all the units responding to the IV change, representing the largest subpopulation for which treatment effects can be identified without further assumptions. The weights are proportional to both the magnitude of the treatment change and the density of X.

4 Estimation and Inference

In this section, we present a semi-parametric approach to estimation and inference for the general case with covariates. The case without covariates, corresponding to having an empty covariate set, is a special instance of this framework and is detailed in Section S.6 in the Appendix for brevity. Our approach assumes that the variables of interest - the instrument and the treatment - are modeled nonparametrically in the specifications for T and Y, respectively, while covariates are incorporated linearly as control variables. This semi-parametric approach is motivated by practical considerations: although fully nonparametric estimation and inference are theoretically viable, they are often computationally prohibitive. Toward the end of this section, we briefly discuss nonparametric inference and examine the practical implications of the additional functional form assumptions required in our method.

4.1 Estimation

We assume a linear quantile regression model for the conditional υ quantile of T given $Z = z$ and $X = x$, i.e., $q_z(x, \upsilon) = a_0(\upsilon) + x'a_1(\upsilon) + za_2(\upsilon) + zx'a_3(\upsilon)$; we further assume a partially linear model for the conditional mean function of Y given Z, X and T, i.e., $m_z(x, t) = x'b_0 + g_0(t) + zx'b_1 + zg_1(t)$, where g_z, $z = 0, 1$, are some unknown functions. Given a sample of $i.i.d.$ observations $\{(Y_i, T_i, X_i, Z_i)\}_{i=1}^n$ for (Y, T, X, Z), we propose the following estimation procedure.

Step 1. Estimate the first-stage conditional treatment quantiles $q_z(x, \upsilon)$:

- $\widehat{q}_z(x, \upsilon) = \widehat{a}_0(\upsilon) + x'\widehat{a}_1(\upsilon) + z\hat{a}_2(\upsilon) + zx'\widehat{a}_3(\upsilon)$

  for $\upsilon \in V^{(l)}$, where $V^{(l)} = \{\upsilon_1, \upsilon_2, ..., \upsilon_l\}$ is the set of equally spaced quantiles over $(0, 1)$. Then $\Delta\widehat{q}(x, \upsilon) = \hat{a}_2(\upsilon) + x'\widehat{a}_3(\upsilon)$.

Step 2. Estimate the conditional mean function $m_z(x,t)$ by a partially linear series estimator:

- $\widehat{m}_z(x,t) = x'\widehat{b}_0 + \widehat{g}_0(t) + zx'\widehat{b}_1 + z\widehat{g}_1(t)$

  Let $\Delta\hat{m}(X_i, v) = \widehat{m}_1(X_i, \widehat{q}_1(X_i, v)) - \widehat{m}_0(X_i, \widehat{q}_0(X_i, v))$.

Step 3. For $v \in V^{(l)}$ and $i = 1, ..., n$, the plug-in estimator of $\pi(X_i, v)$ is $\hat{\pi}(X_i, v) = \Delta\hat{m}(X_i, v)/\Delta\widehat{q}(X_i, v)$.

- Estimate $\pi(v)$: $\widehat{\pi}(v) = \sum_i \widehat{\pi}(X_i, v)\widehat{\omega}_v(X_i)$, where $\widehat{\omega}_v(X_i) = \frac{|\Delta\widehat{q}(X_i,v)|}{\sum_i |\Delta\widehat{q}(X_i,v)|}$
- Estimate π^{DR}: $\widehat{\pi}^{DR} = \sum_{v\in V^{(l)}} \sum_i \widehat{\pi}(X_i, v)\widehat{\omega}(X_i, v)$, where $\widehat{\omega}(X_i, v) = \frac{|\Delta\widehat{q}(X_i,v)|}{\sum_{v\in V^{(l)}}\sum_i |\Delta\widehat{q}(X_i,v)|}$.

The following provides details on the partial linear series estimator in Step 2. Let $\{\psi_{J1}, ..., \psi_{JJ}\}$ be a collection of basis functions of t for approximating the nonparametric component $g_z(t)$. Let $\psi^J(x,t,z) = \left(x', \psi_{J1}(t), ..., \psi_{JJ}(t), zx', z\psi_{J1}(t), ..., z\psi_{JJ}(t)\right)'$, a $2(d_x + J) \times 1$ vector. Let $\Psi = (\psi^J(X_1, T_1, Z_1), ..., \psi^J(X_n, T_n, Z_n))'$, a $n \times 2(d_x + J)$ matrix. Then the series coefficient estimate is $\hat{c} = (\Psi'\Psi)^{-1}\Psi'(Y_1, ..., Y_n)'$, and a series least squares estimator of $m_z(x,t)$ is $\hat{m}_z(x,t) = \psi^J(x,t,z)'\hat{c}$.

Note that the estimand for π^{DR} in Proposition 2 is essentially a ratio of average, i.e.,

$$\pi^{DR} = \frac{\int_0^1 \int_{\mathcal{X}} \Delta m(x,v)\mathrm{sgn}(\Delta q(x,v)) f_X(x)dxdv}{\int_0^1 \int_{\mathcal{X}} \Delta q(x,v)\mathrm{sgn}(\Delta q(x,v)) f_X(x)dxdv},$$

where the sign $\mathrm{sgn}(\Delta q(x,v)) = 1(\Delta q(x,v) > 0) - 1(\Delta q(x,v) < 0)$. Correspondingly, our estimator $\widehat{\pi}^{DR} = \sum_{v\in V^{(l)}} \sum_i \Delta\hat{m}(X_i, v)\mathrm{sgn}(\Delta\hat{q}(X_i, v))/\sum_{v\in V^{(l)}} \sum_i \Delta\hat{q}(X_i, v)\mathrm{sgn}(\Delta\hat{q}(X_i, v))$. $\hat{\pi}^{DR}$ does not involve trimming, which would drop small $|\Delta\hat{q}(X_i, v)|$. Our asymptotic theory characterizes the first-order influence of the Step 1 quantile regression in determining the sign of $\Delta q(X_i, v)$ from the weights. This is different from the regression discontinuity design in Dong, Lee, and Guo (2023), where the estimation is fully nonparametric locally around the cutoff and uses a trimming parameter to control the influence to be of smaller order.

4.2 Inference

This section presents inference results for $\pi(\upsilon)$ and π^{DR}. Inference results for the other parameters $\pi(x, \upsilon)$ and alternative DR estimands defined over $\mathcal{S}_+ = \{(x, \upsilon) \in \mathcal{S}\colon \Delta q(x, \upsilon) > 0\}$ or $\mathcal{S}_- = \{(x, \upsilon) \in \mathcal{S}\colon \Delta q(x, \upsilon) < 0\}$ are presented in Section S.4 and Section S.5, respectively, in the Appendix.

We derive the asymptotic theory based on the literature of quantile regression and sieve estimation. The main complication here is that we need to account for the variation from the Step 1 quantile regression and Step 2 sieve estimation, as well as the sign function. Let $a(\upsilon) = (a_0(\upsilon), a_1'(\upsilon), a_2(\upsilon), a_3'(\upsilon))'$ be the quantile coefficients in Step 1. For the quantile regression estimator $\hat{a}(\upsilon)$, we apply the results of Angrist, Chernozhukov, and Fernández-Val (2006). They show that $\hat{a}(\upsilon)$ converges uniformly over υ in a closed subset of $(0, 1)$ to a zero mean Gaussian process indexed by υ. For the partially linear estimation in Step 2, we apply the results of Chen and Christensen (2018). They establish uniform inference for nonlinear functionals of nonparametric IV regression. We apply their results for a special case of exogenous regressors and linear functionals. Our assumptions for asymptotics collect the assumptions in these two papers. To save space, we list these assumptions in Section S.3 in the Appendix.

We show in Theorem 3 below that under Assumptions A1, A2, and A3, the influence function of $\hat{\pi}(\upsilon)$ is given by $R_i(\upsilon)/B(\upsilon) = (R_{1i}(\upsilon) + R_{2i}(\upsilon) + R_{3i}(\upsilon))/B(\upsilon)$, where $R_{1i}(\upsilon)$ captures the impact of Step 1, $R_{2i}(\upsilon)$ captures the impact of Step 2, $R_{3i}(\upsilon)$ is the influence function for the sample analogue estimator of $\hat{\pi}(\upsilon)$ (without accounting for the step 1 and step 2 estimation errors) in Step 3, and $B(\upsilon)$ is from the normalization in the weighting function. The exact formulas of $R_{ki}(\upsilon)$, $k = 1, 2, 3$, are given in (S.11) in the Appendix. Let $\sigma_n^2(\upsilon) = \mathbb{E}\left[R_i(\upsilon)^2\right]/B(\upsilon)^2$, which is the sieve variance of $\sqrt{n}\hat{\pi}(\upsilon)$. Further let $\hat{\sigma}^2(\upsilon)$ be a uniformly consistent estimator of $\sigma_n^2(\upsilon)$ in the sense that $\sup_{\upsilon\in\mathcal{V}} |\sigma_n(\upsilon)/\hat{\sigma}(\upsilon) - 1| = o_p(1)$ for a closed set $\mathcal{V} = \{\upsilon \ \ \Pr(|\Delta q(X, \upsilon)| > 0) > 0\}$. For example, $\hat{\sigma}^2(\upsilon)$ can be estimated by the sample analogue plug-in estimator, i.e., $\hat{\sigma}^2(\upsilon) = n^{-1}\sum_{i=1}^n \widehat{R}_i(\upsilon)^2/\widehat{B}(\upsilon)^2$, where $\widehat{R}_i(\upsilon)$ and $\widehat{B}(\upsilon)$ are uniformly consistent estimators of $R_i(\upsilon)$ and $B(\upsilon)$, respectively. We give the estimation detail of $\hat{\sigma}^2(\upsilon)$ in Section S.7 in the Appendix.

Theorem 3. *Let Assumptions A1, A2, and A3 hold. Then uniformly for* $\upsilon \in \mathcal{V}$, $\sqrt{n}\big(\hat{\pi}(\upsilon) - \pi(\upsilon)\big)/\hat{\sigma}(\upsilon) = n^{-1/2}\sum_{i=1}^{n} R_i(\upsilon)/(B(\upsilon)\sigma_n(\upsilon)) + o_p(1) \xrightarrow{d} \mathcal{N}(0,1)$.

A $100(1-\alpha)\%$ confidence interval for $\pi(\upsilon)$ can be constructed as $\big[\hat{\pi}(\upsilon) - z^*_{1-\alpha}\hat{\sigma}(\upsilon)/\sqrt{n}, \hat{\pi}(\upsilon) + z^*_{1-\alpha}\hat{\sigma}(\upsilon)/\sqrt{n}\big]$, where $z^*_{1-\alpha} = \Phi^{-1}(1-\alpha/2)$ is the $1-\alpha/2$ quantile of the standard normal distribution, based on the asymptotically normal approximation.

Similarly, Theorem 4 shows that under Assumptions A1, A2, and A3, the influence function of $\hat{\pi}^{DR}$ is given by $R_i/B = (R_{1i} + R_{2i} + R_{3i})/B$. The exact formulas of R_{ki}, $k = 1, 2, 3$, are given in (S.10) in the Appendix. Let $\sigma_n^2 = \mathbb{E}\left[R_i^2\right]/B^2$, which is the sieve variance of $\sqrt{n}\hat{\pi}^{DR}$. Further let $\hat{\sigma}^2$ be a consistent estimator of σ_n^2 such that $|\sigma_n/\hat{\sigma} - 1| = o_p(1)$.

Theorem 4. *Let Assumptions A1, A2, and A3 hold. Let* $\sqrt{n}l^{-1} = o(1)$. *Then* $\sqrt{n}\big(\hat{\pi}^{DR} - \pi^{DR}\big)/\hat{\sigma} = n^{-1/2}\sum_{i=1}^{n} R_i/(B\sigma_n) + o_p(1) \xrightarrow{d} \mathcal{N}(0,1)$.

Based on Theorem 4, a $100(1-\alpha)\%$ confidence interval for π^{DR} can be constructed as $\big[\hat{\pi}^{DR} - z^*_{1-\alpha}\hat{\sigma}/\sqrt{n}, \hat{\pi}^{DR} + z^*_{1-\alpha}\hat{\sigma}/\sqrt{n}\big]$.

Our semi-parametric estimation relies on certain functional form assumptions. The causal interpretation of the estimated parameters depends on the validity of these assumptions. In theory, fully nonparametric estimation and inference are feasible. For instance, in Step 1, the nonparametric QR series method from Belloni et al. (2009) could be employed, and in Step 2, the fully nonparametric mean regression approach from Chen and Christensen (2018) could be applied. Our asymptotic theory for $\hat{\pi}(\upsilon)$ and $\hat{\pi}$ can be extended to these corresponding nonparametric estimators, albeit at the cost of more complex notation and stronger regularity conditions.

When weak monotonicity (along with other identifying assumptions) holds, $\pi^{DR} = \tau^{Wald\_X}$. This implies that if the assumed semi-parametric functional forms are correct, or if both estimators are obtained nonparametrically, they will converge to the same causal parameter. Consequently, their estimates should be similar in large samples. Substantial differences between the two estimates may indicate a violation of monotonicity, assuming the other identifying assumptions hold. Notably, when monotonicity does not hold, the usual Wald estimator becomes inconsistent, whereas our estimator remains consistent for a well-defined causal parameter.

5 Extensions to a Multi-valued IV or Multiple Discrete IVs

In this section, we briefly explore extensions of identification, estimation, and inference to scenarios involving a multi-valued instrumental variable (IV) or a vector of discrete IVs.[8] We begin by examining the basic setup without covariates, followed by a discussion of the more general framework that includes covariates.

Assume $T = g(T, \varepsilon)$ and $T = h(Z, U)$ as in Section 2. Denote the support of Z as $\mathcal{Z} = \{z_0, z_1, ..., z_K\}$. So e.g., if $Z = (Z_1, Z_2)$, where $Z_1 \in \{0, 1\}$ and $Z_2 \in \{0, 1\}$, then one can let $z_0 = (0, 0)$, $z_1 = (0, 1)$, $z_2 = (1, 0)$, and $z_3 = (1, 1)$. Let $U_k = F_{T_{z_k}}\left(T_{z_k}\right)$ be the rank of the potential treatment T_{z_k} if Z is exogenously set to be z_k. The observed rank can be written as $U = \sum_{k=1}^{K} 1(Z = z_k) U_k$. Let $T_{z_k}(u)$ be the u quantile of the potential treatment T_{z_k}. Further let $r_k = \Pr(Z = z_k)$, $p(Z) = \mathbb{E}[T|Z]$, $p_k = \mathbb{E}[T|Z = z_k]$, and $\overline{p} = \mathbb{E}[T]$. Without loss of generality, assume that the $K + 1$ values of Z is ordered such that $p_k \geq p_{k-1}$ for $k = 1, ..., K$, which may involve rearranging and is verifiable from the data.

We continue to use the same sets of assumptions when we consider either the basic setup without covariates or the general setup with covariates, except that the relevant assumptions need to be modified to accommodate the greater support of Z, which is $\mathcal{Z} = \{z_0, z_1, ..., z_K\}$. For example, Assumption 1 now requires that $T_{z_k}(u)$ is strictly monotonic in u for any $z_k \in \mathcal{Z}$, and that $U_k \sim Unif(0, 1)$ for $k = 0, ..., K$, and Assumption 2 independence now requires $Z \perp (U_k, \varepsilon)$ for $k = 0, ..., K$. The same holds true for Assumptions C1 and C2. Further Assumptions 3, 4, and 5, and similarly Assumptions C3, C4 and C5 need to hold for each pair of IV values z_k and z_{k-1} for $k = 1, ..., K$. That is, Assumption 3 monotonicity now states that $\Pr\left(T_{z_k} \geq T_{z_{k-1}}\right) = 1$, $k = 1, ..., K$, while Assumption 4 now requires that $T_{z_k}(u) \neq T_{z_{k-1}}(u)$ for $k = 1, ..., K$ and a set of $u \in (0, 1)$ with a straight positive measure. Assumption 5 now requires that $U_k|\varepsilon \sim U_{k-1}|\varepsilon$, $k = 1, ..., K$. The same holds true for Assumptions C3, C4 and C5. In addition, Assumption C6

[8] Mogstad et al. (2021) show that the LATE monotonicity may not be plausible with multiple IVs for a binary treatment. This conclusion is generalizable to a continuous treatment. While they seek to provide a causal interpretation for the usual two stage least square (2SLS) estimand under a weaker partial monotonicity condition (i.e., monotonicity holds with one IV while holding other IVs fixed), we provide an estimand that is robust to the failure of the LATE monotonicity assumption.

common support now requires $\Pr\left(Z=z_k|X=x\right)\in(0,1)$ for $k=0,...,K$ and any $x\in\mathcal{X}$.

Define the following estimand for each pair of the IV values $\{z_{k-1},z_k\}$, $k=1,...K$,

$$\tau_k(u):=\frac{\mathbb{E}\left[Y|Z=z_k,U=u\right]-\mathbb{E}\left[Y|Z=z_{k-1},U=u\right]}{\mathbb{E}\left[T|Z=z_k,U=u\right]-\mathbb{E}\left[T|Z=z_{k-1},U=u\right]}$$

if the denominator is not zero; otherwise, define $\tau_k(u):=0$. Like before, T and U follow a one-to-one mapping given $Z=z_k$, so conditioning on $U=u$ is the same as conditioning on $T=T_{z_k}(u)$. Further given $Z\perp(U_k,\varepsilon)$, we have $T_{z_k}(u)=q_k(u)$, where $q_k(u)=F^{-1}_{T|Z}\left(u|z_k\right)$ is the conditional u quantile of T given $Z=z_k$. Then $\tau_k(u)$ can be re-written as

$$\tau_k(u)=\frac{\mathbb{E}\left[Y|Z=z_k,T=q_k(u)\right]-\mathbb{E}\left[Y|Z=z_{k-1},T=q_{k-1}(u)\right]}{q_k(u)-q_{k-1}(u)}.$$

Following Theorem 1, $\tau_k(u)$ identifies an average treatment effect at the u quantile of treatment for units responding to the IV change from z_{k-1} to z_k.

Analogous to Proposition 1, define a DR estimand for each pair of IV values. In particular, let $\Delta q_k(u)=q_k(u)-q_{k-1}(u)$, $k=1,...,K$. The corresponding DR estimand is given by

$$\tau_k^{DR}:=\int_0^1\tau_k(u)w_k\left(u\right)du,$$

where $w_k(u)=\frac{|\Delta q_k(u)|}{\int_0^1|\Delta q_k(u)|du}$. τ_k^{DR} identifies a weighted average of the average treatment effect for all units that respond to the IV change from z_{k-1} to z_k, under either monotonicity or rank similarity. Construct an aggregated DR estimand as

$$\tau^{DR,K}:=\sum_{k=1}^{K}\lambda_k\tau_k^{DR}, \tag{16}$$

where $\lambda_k:=\frac{(p_k-p_{k-1})\sum_{l=k}^K r_l\,(p_l-\overline{p})}{\sum_{k=1}^K(p_k-p_{k-1})\sum_{l=k}^K r_l\,(p_l-\overline{p})}$. The weights λ_k follow from Theorem 2 of Imbens and Angrist (1994).

Note that $\lambda_k\geq 0$ and $\sum_{k=1}^K\lambda_k=1$, because the IV values are ordered such that $p_k\geq p_{k-1}$.

Therefore, $\tau^{DR,K}$ is a convex combination of τ_k^{DR}, $k = 1, ..., K$, and hence has the DR property as well.[9] In particular, when monotonicity holds, τ_k^{DR} reduces to the LATE Wald ratio $\tau_k^{Wald}:=\frac{\mathbb{E}[Y|Z=z_k]-\mathbb{E}[Y|Z=z_{k-1}]}{\mathbb{E}[T|Z=z_k]-\mathbb{E}[T|Z=z_{k-1}]}$, and hence $\tau^{DR,K} = \sum_{k=1}^{K} \lambda_k \tau_k^{Wald}$. Further by Theorem 2 of Imbens and Angrist (1994), $\sum_{k=1}^{K} \lambda_k \tau_k^{Wald} = \frac{Cov(Y,p(Z))}{Cov(T,p(Z))}$. Notice that τ_k^{Wald} in this case identifies a weighted average of LATEs for $Z \in \{z_{k-1}, z_k\}$ under monotonicity. Therefore, if monotonicity holds, $\tau^{DR,K}$ identifies a doubly weighted average of LATEs for different compliers, averaging over different compliers for a given pair of IV values and over different pairs of IV values; otherwise, when rank similarity holds, $\tau^{DR,K}$ identifies a doubly weighted average of the average treatment effects at different treatment quantiles - the first averaging is over different treatment quantiles for a given pair of IV values and the second is over different pairs of IV values. Either way, $\tau^{DR,K}$ identifies a doubly weighted average of the average treatment effects for all the units responding to IV changes.

Now consider the general setup where the IV independence and treatment rank similarity are valid only conditional on covariates. One can incorporate covariates as before for each pair of IV values. In particular for $k = 1, ..., K$, define the following estimand

$$\pi_k(x,v):=\frac{\mathbb{E}[Y|Z=z_k, X=x, V=v]-\mathbb{E}\left[Y|Z=z_{k-1}, X=x, V=v\right]}{\mathbb{E}[T|Z=z_k, X=x, V=v]-\mathbb{E}\left[T|Z=z_{k-1}, X=x, V=v\right]}$$

when the denominator is not zero; define $\pi_k(x,v):=0$, otherwise. Following Theorem 2, $\pi_k(x,v)$ identifies an average treatment effect at the conditional (on $X = x$) v quantile of treatment.

Further analogous to Proposition 2, define the DR estimand for each pair of IV values, z_{k-1} and z_k, as

$$\pi_k^{DR}:=\iint_{(0,1)\times\mathcal{X}} \pi_k(x,v) w_k(x,v) dv dx, \tag{17}$$

where $w_k(x,v) = \frac{|\Delta q_k(x,v)| f(x)}{\iint_{(0,1)\times\mathcal{X}} |\Delta q_k(x,v)| f(x) dv dx}$, and $\Delta q_k(x,v) = q_k(x,v) - q_{k-1}(x,v)$, and $q_k(x,v) = F_{T|Z,X}^{-1}(v|z_k, x)$ is the conditional v quantile of T given $Z = z_k$ and $X = x$.

[9]In theory, any convex combination of $\tau_{z_k,z_{k-1}}^{DR}$, $k = 1, ..., K$, would have the DR property. Here our goal is to incorporate the 2SLS or LATE-type estimand given by $\frac{Cov(Y,p(Z))}{Cov(T,p(Z))}$ as a special case, which leads to the particular choice of λ_k.

Then define the aggregated DR estimand as

$$\pi^{DR,K} := \sum_{k=1}^{K} \lambda_k \pi_k^{DR},$$

where λ_k is defined as in (16). When conditional monotonicity holds, $\pi^{DR,K}$ identifies a doubly weighted average of LATEs for compliers and definers; otherwise when rank similarity holds, $\pi^{DR,K}$ identifies a doubly weighted average of the average treatment effects at different conditional treatment quantiles. Note that the identified parameter in this case is still the unconditional doubly weighted average, even though the instrument validity holds only conditional on covariates.

One can estimate $\pi^{DR,K}$ by $\hat{\pi}^{DR,K} = \sum_{k=1}^{K} \hat{\lambda}_k \hat{\pi}_k^{DR}$ given an $i.i.d.$ sample $\{(Y_i, T_i, X_i, Z_i)\}_{i=1}^{n}$, where $\hat{\pi}_k^{DR}$ is an estimator of π_k^{DR} and $\hat{\lambda}_k$ is an estimator of λ_k. $\hat{\pi}_k^{DR}$ can be obtained similar to $\widehat{\pi}^{DR}$ proposed for a binary IV. $\hat{\lambda}_k$ can be estimated by a simple sample analogue plug-in estimator. Let $D^k = 1(Z = z_k)$. One can estimate $p_k = \mathbb{E}[T|Z = z_k]$ by $\widehat{p}_k = \sum_{i=1}^{n} T_i D_i^k / \sum_{i=1}^{n} D_i^k$ for $k = 0, 1, ..., K$, and estimate $\overline{p}$ by $\widehat{\overline{p}} = n^{-1} \sum_{i=1}^{n} T_i$. One can further estimate r_k by $\widehat{r}_k = n^{-1} \sum_{i=1}^{n} D_i^k$ for $k = 1, ..., K$. Then the plug-in estimator for λ_k is $\hat{\lambda}_k = \frac{(\widehat{p}_k - \widehat{p}_{k-1}) \sum_{l=k}^{K} \widehat{r}_l (\widehat{p}_l - \widehat{\overline{p}})}{\sum_{k=1}^{K} (\widehat{p}_k - \widehat{p}_{k-1}) \sum_{l=k}^{K} \widehat{r}_l (\widehat{p}_l - \widehat{\overline{p}})}$, $k = 1, ..., K$

We provide the influence function for $\hat{\pi}^{DR,K}$, denoted as R_{Ki}, in eq. (S.15) in the Appendix. The influence function given in Theorem 4 is now indexed by k, i.e., R_i/B defined in (S.10) is now R_i^k/B^k. Together with the influence function of $\hat{\lambda}_k$, we can derive the influence function of $\hat{\pi}^{DR,K}$. Define the sieve variance of $\sqrt{n}\hat{\pi}^{DR,K}$ as $\sigma_{Kn}^2 = \mathbb{E}\left[R_{Ki}{}^2\right]$. Let $\hat{\sigma}_K^2$ be a consistent estimator of σ_{Kn}^2, such that $|\sigma_{Kn}/\hat{\sigma}_K - 1| = o_p(1)$. We have the following asymptotics result for $\hat{\pi}^{DR,K}$.

Theorem 5. *Let the conditions in Theorem 4 hold. Then* $\sqrt{n}\left(\hat{\pi}^{DR,K} - \pi^{DR,K}\right)/\hat{\sigma}_K$
$= n^{-1/2} \sum_{i=1}^{n} R_{Ki}/\sigma_{Kn} + o_p(1) \xrightarrow{d} \mathcal{N}(0, 1)$.

6 Empirical Application

In this section, we apply the proposed estimators to estimate the effects of night sleep on physical and psychological well-being, using data from a recent field experiment (Bessone et al., 2021). The experiment involved 452 adults in Chennai, India, and spanned a 28-day period. During the first eight days, baseline data were collected. Participants were then randomized into three groups: a control group, an Encouragement group, provided with (a) devices to improve their home sleep environments and (b) information and verbal encouragement to increase night sleep, and an Encouragement + Incentives group, provided with (a) and (b), as well as additional financial incentives to increase night sleep. These groups were further cross-randomized with a nap assignment, offering participants a daily half-hour afternoon nap at their workplace. This resulted in six experimental groups: control, encouragement, encouragement + incentives, naps, encouragement and naps, and encouragement + incentives and naps. Bessone et al. (2021) primarily focused on the reduced-form impacts of these assignments on sleep duration, work outcomes, well-being, cognitive measures, and economic preferences.

For our analysis, we use data from the first three groups (control, encouragement, and encouragement + incentives) without nap assignments. We focus on the well-being index as the outcome variable and night sleep (in hours) as the treatment variable for instrumental variable (IV) analysis. The well-being index is a composite measure of physical and psychological well-being.[10] The well-being index is standardized using the baseline control group's mean and standard deviation, following Bessone et al. (2021). Hence, its unit of measurement is standard deviations. Night sleep is our focus for two reasons: (1) night sleep is the primary form of sleep for most individuals, and (2) the control group reports zero hours of nap, and our treatment variable must be absolutely continuous. As in Bessone et al. (2021), our analysis controls for baseline measures of well-being and night sleep. In some analyses, we also control for participants' gender and age in quartiles.

Our sample comprises 226 observations: 77 from the control group, 75 from the Encourage-

[10] The well-being index is constructed as a weighted average of standardized measures of psychological and physical well-being. For details, see Bessone et al. (2021).

Table 1: Sample summary statistics

	(1)	(2)	(3)	$(2)-(1)$	$(3)-(1)$
Baseline well-being	0.00 (0.46)	0.03 (0.40)	0.09 (0.41)	0.03 (0.07)	0.19 (0.07)
Baseline night sleep	5.51 (0.90)	5.60 (0.84)	5.65 (0.79)	0.09 (0.14)	0.14 (0.14)
Age in 1st quartile	0.23 (0.43)	0.25 (0.44)	0.31 (0.47)	0.02 (0.07)	0.08 (0.07)
Age in 2nd quartile	0.27 (0.45)	0.27 (0.45)	0.20 (0.40)	-0.01 (0.07)	-0.07 (0.07)
Age in 3rd quartile	0.23 (0.43)	0.27 (0.45)	0.34 (0.48)	0.03 (0.07)	0.10 (0.07)
Female	0.68 (0.47)	0.64 (0.48)	0.64 (0.48)	-0.04 (0.08)	-0.04 (0.08)
Night sleep	5.62 (0.80)	5.99 (0.85)	6.22 (0.95)	0.37 (0.14)	0.60 (0.14)
Well-being	-0.00 (0.41)	0.14 (0.37)	0.10 (0.37)	0.15 (0.06)	0.10 (0.06)
Participants	77	75	74		

Note: Columns 1 - 3 report sample means and standard deviations (in parentheses) of the three groups: (1) Control, (2) Encouragement, (3) Encouragement + Incentives ; Columns 4 and 5 report the mean differences and their standard errors.

ment group, and 74 from the Encouragement + Incentives group. Summary statistics are presented in Table 1, which show that the three experimental groups are well-balanced across all covariates.

Consistent with Bessone et al. (2021), assignment to either the Encouragement or Encouragement + Incentives group significantly increases average sleep duration at night, with the Encouragement + Incentives group showing a larger increase. Interestingly, simple mean comparisons reveal that assignment to the Encouragement group significantly improves well-being (by 0.15 standard deviations). In contrast, assignment to the Encouragement + Incentives group does not have a significant impact on well-being, despite leading to a larger average increase in night sleep (0.60 vs. 0.37 hours).

To examine the causal effects of night sleep on well-being, we define two binary IVs: Z_1 is an indicator for assignment to the Encouragement group, and Z_2 be an indicator for assignment to the Encouragement + Incentives group. We investigate the distributional changes in night sleep using each IV and assess the effects of night sleep on well-being across different quantiles of the sleep distribution. We implement the estimator $\hat{\pi}(v)$ as decribed in Section 4.1, which is based on eq. (14). Bootstrapped standard errors are calculated using 1,000 replications for computational efficiency and convenience.

Results using Z_1 as an IV are presented in Table 2, while results using Z_2 are in Table 3. When Z_1 is used, the analysis compares the Encouragement group to the control group. When Z_2 is used, it compares the Encouragement + Incentives group to the control group. Importantly, these

IV analyses condition on the other IV being zero, which is important as discussed in Mogstad et al. (2021).

Table 2 Effects of night sleep on well-being at different conditional quantiles of sleep (IV: Encouragement vs. Control)

Quantile	Sleep (hrs), control	Avg. first-stage sleep change	Sleep effect
0.1	4.53	0.307 (0.088)***	0.276 (0.175)
0.15	4.83	0.295 (0.081)***	0.311 (0.184)*
0.2	5.01	0.226 (0.079)***	0.135 (0.182)
0.25	5.19	0.261 (0.082)***	0.423 (0.184)**
0.3	5.34	0.285 (0.079)***	0.402 (0.188)**
0.35	5.41	0.309 (0.081)***	0.314 (0.163)*
0.4	5.53	0.280 (0.079)***	0.318 (0.158)**
0.45	5.57	0.260 (0.080)***	0.321 (0.169)*
0.5	5.62	0.318 (0.079)***	0.286 (0.179)
0.55	5.76	0.400 (0.083)***	0.319 (0.179)*
0.6	5.86	0.418 (0.080)***	0.270 (0.193)
0.65	5.96	0.330 (0.083)***	0.355 (0.227)
0.7	6.04	0.363 (0.087)***	0.286 (0.227)
0.75	6.23	0.347 (0.083)***	0.225 (0.224
0.8	6.33	0.358 (0.095)***	0.168 (0.229
0.85	6.47	0.340 (0.124)***	0.106 (0.210)
0.9	6.63	0.561 (0.140)***	0.073 (0.193)

Note: This table reports (in column 2) the quantiles of night sleep in hours for the control group, (in column 2) the first-stage quantile changes at different conditional quartiles of night sleep and (in Column 3) the weighted average of average treatment effects at different conditional quantiles of night sleep (estimated based on eq. (14)). Covariates conditioned on are baseline well-being, baseline night sleep, participants' gender and age in four quartiles. The binary IV is Z_1, the Encouragement group vs. Control. Bootstrapped standard errors (based on 1,000 replications) are in the parenthesis. *** Significant at 1%; ** Significant at 5%; * Significant at 10%.

Estimates in Table 2 show significant increases in night sleep at all quantiles, ranging from 0.23 to 0.56 hours. However, significant positive effects of night sleep on well-being are observed only at lower quantiles, with the effects diminishing at higher quantiles.

Table 3 similarly shows significant increases in night sleep across all quantiles, with larger increases compared to Table 2. However, these additional increases in night sleep do not translate into improved well-being. The estimated effects on well-being are smaller at nearly all quantiles, statistically insignificant, and decrease with increasing quantiles, eventually becoming negative. These findings align with the notion of diminishing returns to night sleep.

Table 3 Effects of per hour night sleep on well-being at different levels of sleep (IV: Encouragement + Incentives vs. Control)

Quantile	Sleep (hrs), control	Average first-stage sleep change	Sleep effect
0.1	4.53	0.405 (0.097)***	0.099 (0.150)
0.15	4.83	0.445 (0.100)***	0.107 (0.164)
0.2	5.01	0.337 (0.099)***	0.162 (0.162)
0.25	5.19	0.347 (0.100)***	0.157 (0.161)
0.3	5.34	0.364 (0.096)***	0.135 (0.152)
0.35	5.41	0.505 (0.097)***	0.105 (0.146)
0.4	5.53	0.523 (0.101)***	0.101 (0.148)
0.45	5.57	0.476 (0.102)***	0.101 (0.150)
0.5	5.62	0.512 (0.094)***	0.100 (0.150)
0.55	5.76	0.654 (0.097)***	0.095 (0.150)
0.6	5.86	0.660 (0.097)***	0.068 (0.146)
0.65	5.96	0.581 (0.098)***	0.055 (0.152)
0.7	6.04	0.586 (0.093)***	0.062 (0.144)
0.75	6.23	0.588 (0.096)***	0.049 (0.139)
0.8	6.33	0.636 (0.105)***	0.037 (0.140)
0.85	6.47	0.601 (0.100)***	0.033 (0.145)
0.9	6.63	0.586 (0.104)***	-0.011 (0.142)

Note: This table reports (in column 2) the quantiles of night sleep in hours for the control group, (in column 2) the first-stage quantile changes at different conditional quartiles of night sleep and (in Column 3) the weighted average of average treatment effects at different conditional quantiles of night sleep (estimated based on eq. (14)). Covariates conditioned on are baseline well-being, baseline night sleep, participants' gender and age in four quartiles. The binary IV is Z_2, the Encouragement + Incentives group vs. Control. Bootstrapped standard errors (based on 1,000 replications) are in the parenthesis. *** Significant at 1%; ** Significant at 5%; * Significant at 10%.

The results in Tables 2 and 3 rely on the assumption of rank similarity, conditional on baseline sleep, baseline well-being, gender, and age. This assumption holds if additional individual-specific determinants of sleep, such as biological preferences or circadian rhythms, act as fixed effects. However, this assumption is untestable with the available data, as existing tests (e.g., Dong and Shen, 2016; Frandsen and Lefgren, 2016) require external "rank shifters" not included as covariates.

We next apply our doubly robust (DR) approach to estimate the weighted average effects among all responding units. Specifically, we estimate the DR estimator $\hat{\pi}_{DR}$ using either Z_1 or Z_2 as a single IV or both IVs in a multi-valued IV analysis. In the first analysis, we compare the Encouragement group to the control group ($Z = Z_1$). In the second, we compare the Encouragement + Incentives group to the control group ($Z = Z_2$). In the third, we include all three groups, defining the IV as $Z = (Z_1, Z_2)$ with $Z \in \{z_0, z_1, z_2\}$, where z_0:=(0, 0), z_1:=(1, 0) and z_2:=(0, 1).

Table 4: Effects of per hour night sleep on well-being, using a single IV

2SLS		Wald		DR	
(I)	(II)	(I)	(II)	(I)	(II)
	IV: Encouragement vs. Control				
0.427	0.408	0.426	0.398	0.391	0.231
(0.195)**	(0.187)*	(0.222)*	(0.170)**	(0.209)*	(0.131)*
	IV: Encouragement + Incentives vs. Control				
0.130	0.111	0.130	0.106	0.123	0.075
(0.109)	(0.107)	(0.109)	(0.104)	(0.128)	(0.116)

Note: 2SLS - linear IV/2SLS estimate; Wald - estimates based on eq. (14) where the conditional mean functions are assumed to be linear in covariates and fully interacted with the binary IV (see details in the main text); DR - doubly robust IV estimates based on the estimation procedure described in Section 4. Columns (I) control for baseline well-being and baseline night sleep; Columns (II) additionally control for participants' gender and age in four quartiles. In the top panel, the binary IV used is Z_1, the indicator for being assigned to the Encouragement group vs. Control; in the bottom panel, the binary IV used is Z_2, the indicator for being assigned to the Encouragement + Incentives group vs. Control. Bootstrapped standard errors (based on 1,000 replications) are in the parentheses. ** Significant at 5%; * Significant at 10%.

Table 4 presents DR estimates, linear 2SLS estimates, and semiparametric Wald estimates (based on τ^{Wald_X} in eq. (15)) using a single IV (Z_1 or Z_2). For the DR estimator, we use a first-order polynomial for the power series of T, given the limited sample size. For the Wald estimator, conditional means are assumed to be linear in covariates, fully interacted with the binary IV. The Wald estimator relies on weaker conditional monotonicity assumptions, whereas the 2SLS estimator requires unconditional monotonicity.[11] In addition, the linear 2SLS estimator requires homogeneity of both the instrument and the treatment effects.

Results in Table 4 indicate that the DR estimates align relatively closely with the Wald and 2SLS estimates. When Z_1 is used, significant positive effects of increased night sleep on well-being are observed, ranging from 0.2 to 0.4 standard deviations. In contrast, the estimates using Z_2 indicate smaller positive effects (around 0.1 standard deviations), which are not statistically significant. The consistency between DR and 2SLS point estimates enhances the credibility of the results.

Table 5: Effects of per hour night sleep on well-being, using two Ivs

	(I)	(II)		(I)	(II)
2SLS	0.151	0.144			
	(0.107)	(0.104)			
Wald	0.181	0.154	DR	0.167	0.099
	(0.217)	(0.157)		(0.180)	(0.113)
$\tau_1^{Wald\_X}$	0.440	0.388	π_1^{DR}	0.400	0.232
	(0.232)	(0.146)***		(0.198)**	(0.136)*
$\tau_2^{Wald\_X}$	-0.331	-0.309	π_2^{DR}	-0.294	-0.166
	(0.345)	(0.211)		(0.189)	(0.125)

Note: All estimates are based on the full sample with 226 observations. Columns (I) control for baseline well-being and baseline night sleep; Columns (II) additionally control for participants' gender and age in four quartiles. π_1^{DR} and $\tau_1^{Wald\_X}$ compare the Encouragement group with the Control group; π_2^{DR} and $\tau_2^{Wald\_X}$ compare the Encouragement + Incentives group with the Encouragement group. DR and Wald estimates are weighted averages of π_1^{DR} and π_2^{DR} or $\tau_1^{Wald\_X}$ and $\tau_2^{Wald\_X}$, respectively, where the weights $\lambda_1 = 0.664$ (std. err. $= 0.225$) and $\lambda_2 = 0.336$ (std. err.$= 0.225$). *** Significant at 1%; ** Significant at 5%; * Significant 10%.

Table 5 reports estimates using Z_1 and Z_2 jointly in a multi-valued IV framework discussed in Section 5. The DR estimates are based on $\pi^{DR,2}:=\sum_{k=1}^{2}\lambda_k\pi_k^{DR}$, where λ_k is defined in

[11] The unconditional monotonicity assumption requires: (a) Everyone increases their night sleep when assigned to the Encouragement group instead of the control group. (b) Everyone increases their night sleep when assigned to the Encouragement + Incentives group instead of the control group. For the multi-valued IV analysis, an additional assumption is that, conditional on covariates, individuals increase their night sleep further when assigned to the Encouragement + Incentives group instead of the Encouragement group. While these assumptions are plausible, they are not directly testable due to small sample sizes and the lack of suitable stochastic dominance tests, particularly when conditioning on covariates. Thus, the doubly robust approach is particularly useful in this context.

eq. (16) and π_k^{DR} in eq. (17). Similar to the single IV analysis, we also estimate the linear 2SLS estimator and the multi-valued IV extension of the Wald estimator. The latter is based on $\tau^{Wald\_X,K} := \sum_{k=1}^{K} \lambda_k \tau_k^{Wald\_X}$ for $K = 2$, with $\tau_k^{Wald\_X}$ defined analogously to τ^{Wald_X} for each pair of IV values, z_{k-1} and z_k for $k = 1, 2$.

The DR and Wald estimates in this case represent weighted averages of the effects for each pair of IV values: π_1^{DR} and $\tau_1^{Wald\_X}$ capture the variation between z_1 and z_0 (Encouragement vs. control); π_2^{DR} and $\tau_2^{Wald\_X}$ capture the variation between z_2 and z_1 (Encouragement + Incentives vs. Encouragement). The overall estimates using both IVs are small, positive, and statistically insignificant. Detailed subgroup analyses reveal positive effects for π_1^{DR} and $\tau_1^{Wald\_X}$, while estimates of π_2^{DR} and $\tau_2^{Wald\_X}$ are consistently negative, though not statistically significant. These findings align with the single IV results, suggesting that the additional sleep induced by financial incentives do not improve well-being.

The above results provide important insights. Individuals with lower baseline sleep levels who moderately increased sleep demonstrated improved mental and physical well-being. In contrast, individuals with higher baseline sleep levels or those incentivized to further increase their sleep did not experience similar benefits.

Our analysis differs from Bessone et al. (2021, Table A.XVII), which uses different IVs jointly in a single regression. By analyzing each IV separately and providing a detailed breakdown of the joint IV estimates, we uncover more nuanced findings about the differential impacts of increased nigh sleep. These results underscore the complexity of well-being outcomes and the need for tailored sleep-related policies.

7 Conclusion

Many empirical applications involve a continuous treatment and a binary or discrete IV. The standard approach, 2SLS, relies on a mean change in the treatment for identifying causal effects. This paper extends the framework by exploring distributional changes in the treatment induced by binary or discrete IVs to identify treatment effects.

We demonstrate that distributional changes in the treatment variable can identify average treatment effects at specific treatment quantiles and weighted averages of these quantile-specific effects. This identification relies on rank restrictions in the first stage, particularly treatment rank invariance or treatment rank similarity. Furthermore, we develop a novel doubly robust (DR) estimand that identifies weighted average effects for all individuals affected by IV changes under either the rank restriction or the standard LATE-type monotonicity assumption.

Building on these nonparametric identification results, we propose semiparametric estimators for treatment effects at different quantiles, capturing heterogeneity across treatment levels. We also introduce a DR estimator for the weighted average treatment effect for all the units responsive to IV changes, ensuring robust identification even when the mean treatment change is zero or when monotonicity does not hold. We establish consistency and asymptotic normality for all the proposed estimators. While our primary focus is on binary IVs, we extend all of the identification, estimation and asymptotic results to cases involving multi-valued IVs or vectors of discrete IVs, with or without covariates.

Our estimators complement standard 2SLS in several key ways. They are particularly useful when 2SLS is either insufficient (e.g., when exploring treatment effect heterogeneity across different treatment levels) or invalid (e.g., when the mean treatment change is zero or when monotonicity does not hold). In such scenarios, the rank similarity assumption becomes crucial. Our framework's generalization to include covariates is especially promising, as conditioning on relevant covariates improves the plausibility of the rank restriction by allowing the first-stage disturbance to be treated as a scalar.

To illustrate the practical utility of our methods, we apply them to evaluate the impact of nighttime sleep on well-being. Our empirical analysis reveals interesting treatment effect heterogeneity in treatment intensity. Furthermore, our DR estimator yields results consistent with traditional 2SLS estimates, thereby enhancing the credibility of the latter.

Future research could extend this framework further by incorporating high-dimensional covariates and leveraging machine learning techniques for estimation, building on our identification results. Such advancements could enhance the flexibility and applicability of our methods in modern empirical settings.

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Online Supplementary Appendix for Doubly Robust Identification of Causal Effects of a Continuous Treatment with Discrete Instruments

Yingying Dong and Ying-Ying Lee

In this supplementary appendix: Section S.1 provides proofs for the identification results presented in Sections 2 and 3. Section S.2 presents the DR estimands defined over subsets of treatment quantiles. Section S.3 presents the detailed assumptions for inference. Section S.4 presents the inference theory for $\pi(x, \upsilon)$. Section S.5 contains the proofs of the inference results presented in Section 4.2. Section S.6 discusses nonparametric estimation and inference without covariates. Section S.7 details the computation of standard errors.

S.1 Proofs: Identification

Proof of Lemma 1: By definition,

$$
\begin{aligned}
\tau^{Wald} &= \frac{\mathbb{E}\left[g(T_1,\varepsilon)|Z=1\right]-\mathbb{E}\left[g(T_0,\varepsilon)|Z=0\right]}{\mathbb{E}[T_1|Z=1]-\mathbb{E}[T_0|Z=0]} \\
&= \frac{\mathbb{E}\left[g(T_1,\varepsilon)-g(T_0,\varepsilon)\right]}{\mathbb{E}[T_1-T_0]} \\
&= \frac{\mathbb{E}\left[\{g(T_1,\varepsilon)-g(T_0,\varepsilon)\}\cdot 1(T_1-T_0>0)\right]}{\mathbb{E}[\{T_1-T_0\}\cdot 1(T_1-T_0>0)]} \\
&= \frac{\iint_{\mathcal{T}_c}\int\{g(t_1,e)-g(t_0,e)\}F_{\varepsilon|T_0,T_1}(de|t_0,t_1)F_{T_0,T_1}(dt_0,dt_1)}{\iint_{\mathcal{T}_c}\{t_1-t_0\}F_{T_0,T_1}(dt_0,dt_1)} \\
&= \iint_{\mathcal{T}_c} w_{t_0,t_1}\left\{\int\frac{g(t_1,e)-g(t_0,e)}{t_1-t_0}F_{\varepsilon|T_0,T_1}(de|t_0,t_1)\right\}F_{T_0,T_1}(dt_0,dt_1) \\
&= \iint_{\mathcal{T}_c} w_{t_0,t_1}\mathbb{E}\left[\frac{Y_{t_1}-Y_{t_0}}{t_1-t_0}|T_0=t_0,T_1=t_1\right]F_{T_0,T_1}(dt_0,dt_1) \\
&= \iint_{\mathcal{T}_c} w_{t_0,t_1}LATE(t_0,t_1)F_{T_0,T_1}(dt_0,dt_1),
\end{aligned}
$$

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where the first equality follows from the models for Y and T without covariates as specified in eq.s (2) and (4), respectively, the second equality follows from Assumption 2, the third equality follows from Assumption 3, the fourth equality follows from the law of iterated expectations, and the fifth to the last equalities follow from rearranging and our notation $w_{t_0,t_1} = \frac{t_1-t_0}{\iint_{\mathcal{T}_c}(t_1-t_0)F_{T_0,T_1}(dt_0,dt_1)}$ and $\mathcal{T}_c = \{(t_0, t_1) \in \mathcal{T}_0 \times \mathcal{T}_1 : t_1 - t_0 > 0\}$. Under monotonicity, $w_{t_0,t_1} \geq 0$ and $\iint_{\mathcal{T}_c} w_{t_0,t_1} F_{T_0,T_1}(dt_0, dt_1) = 1$, so τ^{Wald} identifies a weighted average of $LATE(t_0, t_1) := \mathbb{E}\left[\frac{Y_{t_1}-Y_{t_0}}{t_1-t_0} | T_0 = t_0, T_1 = t_1\right]$ for $(t_0, t_1) \in \mathcal{T}_c$.

Further, when $g(T, \varepsilon)$ is continuously differentiable in T,

$$\begin{aligned}
\tau^{Wald} &= \frac{\mathbb{E}\left[\int_{T_0}^{T_1} \frac{\partial g(t,\varepsilon)}{\partial t} dt\right]}{\mathbb{E}\left[\int_{T_0}^{T_1} 1 dt\right]} \\
&= \frac{\mathbb{E}\left[\int_{\mathcal{T}} \frac{\partial g(t,\varepsilon)}{\partial t} 1\left(T_0 \leq t \leq T_1\right) dt\right]}{\mathbb{E}\left[\int 1\left(T_0 \leq t \leq T_1\right) dt\right]} \\
&= \frac{\int_{\mathcal{T}} \mathbb{E}\left[\frac{\partial g(t,\varepsilon)}{\partial t} | T_0 \leq t \leq T_1\right] \Pr\left(T_0 \leq t \leq T_1\right) dt}{\int \Pr\left(T_0 \leq t \leq T_1\right) dt} \\
&= \int_{\mathcal{T}} \mathbb{E}\left[\frac{\partial g(t,\varepsilon)}{\partial t} \middle| T_0 \leq t \leq T_1\right] \widetilde{\omega} dt,
\end{aligned}$$

where $\widetilde{\omega} = \frac{\Pr(T_0 \leq t \leq T_1)}{\int_{\mathcal{T}} \Pr(T_0 \leq t \leq T_1) dt}$, the first equality follows from Assumption 3 and differentiability of $g(T, \varepsilon)$ in T, the second to the last equalities follow from the law of iterated expectations and interchanging the order of integration when standard regularity conditions hold.

Proof of Lemmas 2 and 3: Assumption 2 states $Z \perp (U_z, \varepsilon)$, which implies $Z \perp U_z|\varepsilon$. That is, $U_0|\varepsilon \sim U_0|(\varepsilon, Z = 0)$ and $U_1|\varepsilon \sim U_1|(\varepsilon, Z = 1)$. Further by Assumption 5, $U_0|\varepsilon \sim U_1|\varepsilon$. Together they imply $U_0|(\varepsilon, Z = 0) \sim U_1|(\varepsilon, Z = 1)$, i.e., $U|(\varepsilon, Z = 1) \sim U|(\varepsilon, Z = 0)$, so that $U \perp Z|\varepsilon$. Further by Assumption 2, $Z \perp \varepsilon$. Therefore, $Z \perp (U, \varepsilon)$, and hence $Z \perp \varepsilon|U$. It further implies $T \perp \varepsilon|U$, since $T = h(Z, U)$.

Replacing the above proof of Lemma 2 by conditioning on X in each step proves Lemma 3.

Proof of Theorem 2: First, one can show $Z \perp \epsilon | (V, X)$ under Assumptions C2 and C5, analogous to the derivation of Lemma 2. Specifically, Assumption C2 states $Z \perp (V_z, \epsilon) | X$, which implies $Z \perp V_z | (X, \epsilon)$, i.e., $V_z | (X, \epsilon, Z = z) \sim V_z | (X, \epsilon)$, and hence $V | (X, \epsilon, Z = z) \sim V_z | (X, \epsilon)$. In addition, Assumption C5 states $V_1 | (X, \epsilon) \sim V_0 | (X, \epsilon)$. Then, $V | (X, \epsilon, Z = 0) \sim V | (X, \epsilon, Z = 1)$, i.e., $Z \perp V | (X, \epsilon)$. Further by Assumption C2, $Z \perp \epsilon | X$. Therefore, $Z \perp (V, \epsilon) | X$, and hence $Z \perp \epsilon | (V, X)$.

Consider now the two terms in the numerator of $\pi (x, v)$:

$$\begin{aligned} \mathbb{E} [Y | Z = z, X = x, V = v] &= \mathbb{E} \left[G (T_z(x, v), x, \epsilon) | Z = z, X = x, V = v \right] \\ &= \mathbb{E} \left[G (T_z(x, v), x, \epsilon) | X = x, V = v \right] \\ &= \mathbb{E} \left[Y_{T_z(x,v)} | X = x, V = v \right] \\ &= \int G (T_z(x, v), x, e) F_{\epsilon | X, V} (de | x, v), \end{aligned}$$

where the first equality follows from our models (9) and (10), the second equality follows from the condition $Z \perp \epsilon | (V, X)$ shown above, and the third equality follows from the definition of potential outcomes.

Consider next the two terms in the denominator of $\pi (x, v)$. By eq. (10),

$$\mathbb{E} [T | Z = z, X = x, V = v] = T_z (x, v).$$

Together they prove the theorem.

Proof of Lemma 4: First notice by definition,

$$\begin{aligned} \Delta Y (x) &= \mathbb{E} [Y | Z = 1, X = x] - \mathbb{E} [Y | Z = 0, X = x] \\ &= \mathbb{E} [G (T_1, X, \epsilon) | Z = 1, X = x] - \mathbb{E} [G (T_0, X, \epsilon) | Z = 0, X = x] \\ &= \mathbb{E} [G (T_1, X, \epsilon) | X = x] - \mathbb{E} [G (T_0, X, \epsilon) | X = x] \\ &= \mathbb{E} [G (T_1, X, \epsilon) - G (T_0, X, \epsilon) | X = x]. \end{aligned} \tag{S.1}$$

where the first equality follows from our models of Y and T, equations (9) and (10), respectively, while the second equality follows from Assumption C2. Similarly,

$$\begin{aligned}\Delta T(x) &= \mathbb{E}[T|Z=1, X=x] - \mathbb{E}[T|Z=0, X=x] \\ &= \mathbb{E}[T_1|Z=1, X=x] - \mathbb{E}[T_0|Z=0, X=x] \\ &= \mathbb{E}[T_1 - T_0|X=x]. \end{aligned} \tag{S.2}$$

Consider first $\tau_0^{Wald\_X}$ under strong monotonicity.

$$\begin{aligned}
&\tau_0^{Wald_X} \\
&= \frac{\int_{\mathcal{X}} \mathbb{E}[G(T_1, X, \epsilon) - G(T_0, X, \epsilon)|X=x] F_X(dx)}{\int_{\mathcal{X}} \mathbb{E}[T_1 - T_0|X=x] F_X(dx)} \\
&= \frac{\mathbb{E}[G(T_1, X, \epsilon) - G(T_0, X, \epsilon)]}{\mathbb{E}[T_1 - T_0]} \\
&= \frac{\mathbb{E}[\{G(T_1, X, \epsilon) - G(T_1, X, \epsilon)\} \cdot 1(T_1 - T_0 > 0)]}{\mathbb{E}[\{T_1 - T_0\} \cdot 1(T_1 - T_0 > 0)]} \\
&= \frac{\iint_{\mathcal{T}_c} \iint \{G(t_1, x, e) - G(t_0, x, e)\} F_{X,\varepsilon|T_0,T_1}(dx, de|t_0, t_1) F_{T_0,T_1}(dt_0, dt_1)}{\iint_{\mathcal{T}_c} \{t_1 - t_0\} F_{T_0,T_1}(dt_0, dt_1)} \\
&= \iint_{\mathcal{T}_c} w_{t_0,t_1} \left\{ \iint \frac{G(t_1, x, e) - G(t_0, x, e)}{t_1 - t_0} F_{X,\varepsilon|T_0,T_1}(dx, de|t_0, t_1) \right\} F_{T_0,T_1}(dt_0, dt_1) \\
&= \iint_{\mathcal{T}_c} w_{t_0,t_1} \mathbb{E}\left[\frac{Y_{t_1} - Y_{t_0}}{t_1 - t_0} | T_0 = t_0, T_1 = t_1\right] F_{T_0,T_1}(dt_0, dt_1) \\
&= \iint_{\mathcal{T}_c} w_{t_0,t_1} LATE^c(t_0, t_1) F_{T_0,T_1}(dt_0, dt_1),
\end{aligned}$$

where $w_{t_0,t_1} = (t_1 - t_0) / \iint_{\mathcal{T}_c} (t_1 - t_0) F_{T_0,T_1}(dt_0, dt_1)$.

Consider now τ^{Wald_X} under the weak conditional monotonicity. Note that conditional monotonicity given in Assumption C3 means that the covariate set $\mathcal{X}$ can be partitioned into non-overlapping subsets $\mathcal{X}^+$ and $\mathcal{X}^-$ such that for $X \in \mathcal{X}^+$, $T_1 \geq T_0$ a.s. and for $X \in \mathcal{X}^-$ $T_1 \leq T_0$ a.s. Further, under Assumption C3, $\mathbb{I}^+(x) = 1 \iff x \in \mathcal{X}^+$ and $\mathbb{I}^-(x) = 1 \iff x \in \mathcal{X}^-$. Let $\mathcal{T}_d = \{(t_0, t_1) \in \mathcal{T}_0 \times \mathcal{T}_1 : t_1 < t_0\}$ be the set of all types of defiers. So $\mathcal{T}_{c,d} = \mathcal{T}_c \cup \mathcal{T}_d$. Consider the numerator of τ^{Wald_X} first.

$$
\begin{aligned}
& \int_{\mathcal{X}} \mathbb{I}^{+}(x)\,\Delta Y(x) - \mathbb{I}^{-}(x)\,\Delta Y(x)\; f_X(x)dx \\
= \;& \int_{\mathcal{X}^{+}} \Delta Y(x)\; f_X(x)dx - \int_{\mathcal{X}^{-}} \Delta Y(x)\; f_X(x)dx \\
= \;& \int_{\mathcal{X}} \{G(T_1, X, \epsilon) - G(T_0, X, \epsilon)\} \cdot 1(T_1 - T_0 > 0)\; f_X(x)dx \\
& + \int_{\mathcal{X}} \{G(T_0, X, \epsilon) - G(T_1, X, \epsilon)\} \cdot 1(T_1 - T_0 < 0)\; f_X(x)dx \\
= \;& \mathbb{E}\left[\begin{array}{c} \{G(T_1, X, \epsilon) - G(T_0, X, \epsilon)\} \cdot 1(T_1 - T_0 > 0) \\ + \{G(T_0, X, \epsilon) - G(T_1, X, \epsilon)\} \cdot 1(T_1 - T_0 < 0) \end{array}\right] \\
= \;& \iint_{\mathcal{T}_c} \iint \{G(t_1, x, e) - G(t_0, x, e)\}\, F_{X,\varepsilon|T_0,T_1}(dx, de|t_0, t_1)\, F_{T_0,T_1}(dt_0, dt_1) \\
& + \iint_{\mathcal{T}_d} \iint \{G(t_0, x, e) - G(t_1, x, e)\}\, F_{\varepsilon|T_0,T_1,X}(dx, de|t_0, t_1)\, F_{T_0,T_1}(dt_0, dt_1),
\end{aligned}
$$

where the first equality follows from that under Assumption C3, $\mathbb{I}^{+}(x) = 1 \Longleftrightarrow x \in \mathcal{X}^{+}$ and $\mathbb{I}^{-}(x) = 1 \Longleftrightarrow x \in \mathcal{X}^{-}$, the second equaltiy follows from eq. (S.1) and that under Assumption C3, $X \in \mathcal{X}^{+}$, $T_1 \geq T_0$ a.s. and for $X \in \mathcal{X}^{-}$ $T_1 \leq T_0$ a.s.

Similarly, the denominator of τ^{Wald_X} can be derived as follows

$$
\begin{aligned}
& \int_{\mathcal{X}} \mathbb{I}^{+}(X)\,\Delta T(x) - \mathbb{I}^{-}(X)\,\Delta T(x)\; f_X(x)dx \\
= \;& \int_{\mathcal{X}} \left\{\mathbb{I}^{+}(X)\,\mathbb{E}[T_1 - T_0|X = x] + \mathbb{I}^{-}(X)\,\mathbb{E}[T_0 - T_1|X = x]\right\} f_X(x)dx \\
= \;& \int_{\mathcal{X}} \mathbb{E}[|T_1 - T_0||X = x]\; f_X(x)dx \\
= \;& E[|T_1 - T_0|] \\
= \;& \iint_{\mathcal{T}_{c,d}} |t_1 - t_0| F_{T_0,T_1}(dt_0, dt_1)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\tau^{Wald\_X} \\
&= \frac{\left(\begin{array}{c} \iint_{\mathcal{T}_c} \iint \{G(t_1, x, e) - G(t_0, x, e)\} F_{X,\varepsilon|T_0,T_1}(dx, de|t_0, t_1)\, F_{T_0,T_1}(dt_0, dt_1) \\ + \iint_{\mathcal{T}_d} \iint \{G(t_0, x, e) - G(t_1, x, e)\} F_{X,\varepsilon|T_0,T_1}(dx, de|t_0, t_1)\, F_{T_0,T_1}(dt_0, dt_1) \end{array} \right)}{\iint_{\mathcal{T}_{c,d}} |t_1 - t_0| F_{T_0,T_1}(dt_0, dt_1)} \\
&= \iint_{\mathcal{T}_c} w_{t_0,t_1} \left\{ \iint \frac{G(t_1, x, e) - G(t_0, x, e)}{t_1 - t_0} F_{X,\varepsilon|T_0,T_1}(dx, de|t_0, t_1) \right\} F_{T_0,T_1}(dt_0, dt_1) \\
&+ \iint_{\mathcal{T}_d} w_{t_0,t_1} \left\{ \iint \frac{G(t_0, x, e) - G(t_1, x, e)}{t_0 - t_1} F_{X,\varepsilon|T_0,T_1}(dx, de|t_0, t_1) \right\} F_{T_0,T_1}(dt_0, dt_1) \\
&= \iint_{\mathcal{T}_c} w_{t_0,t_1} \mathbb{E}\left[\frac{Y_{t_1} - Y_{t_0}}{t_1 - t_0} | T_1 = t_1, T_0 = t_0 \right] F_{T_0,T_1}(dt_0, dt_1) \\
&+ \iint_{\mathcal{T}_d} w_{t_0,t_1} \mathbb{E}\left[\frac{Y_{t_0} - Y_{t_1}}{t_0 - t_1} | T_1 = t_1, T_0 = t_0 \right] F_{T_0,T_1}(dt_0, dt_1) \\
&= \iint_{\mathcal{T}_{c,d}} w_{t_0,t_1} \mathbb{E}\left[\frac{Y_{t_1} - Y_{t_0}}{t_1 - t_0} | T_1 = t_1, T_0 = t_0 \right] F_{T_0,T_1}(dt_0, dt_1) \\
&= \iint_{\mathcal{T}_{c,d}} w_{t_0,t_1} LATE^{c,d}(t_0, t_1) F_{T_0,T_1}(dt_0, dt_1),
\end{aligned}
$$

where recall $LATE^{c,d}(t_0, t_1) = LATE^{c}(t_0, t_1) = \mathbb{E}\left[\frac{Y_{t_1} - Y_{t_0}}{t_1 - t_0} | T_1 = t_1, T_0 = t_0 \right]$ by definition, and $w_{t_0,t_1} = |t_1 - t_0| / \iint_{\mathcal{T}_c} |t_1 - t_0| F_{T_0,T_1}(dt_0, dt_1)$.

S.2 Alternative DR estimands

This section briefly discusses alternative DR estimands defined over subsets of treatment quantiles in which treatment changes are consistently positive or negative.

Consider first the model setup without covariates specified in Section 2. Let $\mathcal{U}_+ = \{u \in \mathcal{U} : \Delta q(u) > 0\}$. Define $\tau_+^{DR} := \int_{\mathcal{U}_+} \tau(u) w_+(u) du$, where $w_+(u) = \Delta q(u) / \int_{\mathcal{U}_+} \Delta q(u)\, du$. τ_+^{DR} can be rewritten as the ratio of the mean outcome difference over $u \in \mathcal{U}_+$ to the mean treatment difference over $u \in \mathcal{U}_+$, i.e.,

$$\tau_+^{DR} = \frac{\int_{\mathcal{U}_+} \int \{g(T_1(u), e) - g(T_0(u), e)\} F_{\varepsilon|U}(de|u)\, du}{\int_{\mathcal{U}_+} \Delta T(u) du}.$$

τ_{+}^{DR} carries a similar interpretation as that of τ^{DR} but is only for the subset of treatment quantiles $u \in \mathcal{U}_{+}$.[12] In particular, when either monotonicity or treatment rank similarity holds over $\mathcal{U}_{+}$, τ_{+}^{DR} identifies a weighted average of the average treatment effects for all the responding (to IV changes) units associated with this subset of quantiles. Similarly, one can define $\tau_{-}^{DR}:=\int_{\mathcal{U}_{-}} \tau(u) w_{-}(u) du$, where $\mathcal{U}_{-}=\{u \in \mathcal{U} : \Delta q(u) < 0\}$ and $w_{-}(u) = \Delta T(u) / \int_{\mathcal{U}_{-}} \Delta q(u)\, du$.

Consider now the more general model setup with covariates given in Section 3. Let $\mathcal{S}_{+} = \{(x, v) \in \mathcal{S} \colon \Delta q(x, v) > 0\}$ and $\mathcal{S}_{-} = \{(x, v) \in \mathcal{S} \colon \Delta q(x, v) < 0\}$. Assume that they are non-empty. Then one can define

$$\pi_{+}^{DR}:=\iint_{\mathcal{S}_{+}} \pi(x, v) w_{+}(x, v) dv dx, \tag{S.3}$$

where $w_{+}(x, v) = \Delta q(x, v) f(x) / \iint_{\mathcal{S}_{+}} \Delta q(x, v) f(x) dv dx$. So π_{+}^{DR} identifies a weighted average of the average treatment effects for all the responding units with $(x, v) \in S_{+}$, when either conditional monotonicity or conditional treatment rank similarity holds for $\mathcal{S}_{+}$. π_{-}^{DR} can be analogously defined by replacing $w_{+}(x, v)$ with $w_{-}(x, v)$ and $\mathcal{S}_{+}$ with $\mathcal{S}_{-}$ in eq. (S.3) respectively. π_{-}^{DR} identifies a weighted average of the average treatment effects for units for all the responding units with $(x, v) \in S_{-}$, regardless of whether they stay at the same treatment rank or not.

For estimation, one may estimate π_{+}^{DR} or π_{-}^{DR} analogously by replacing $|\Delta\widehat{q}(X_i, v)|$ in Section 4.1 with $\Delta\widehat{q}(X_i, v)$ or $-\Delta\widehat{q}(X_i, v)$, respectively.

S.3 Assumptions for Inference

Our inference requires the following assumptions. Assumption A1 collects the conditions in Theorem 3 in Angrist, Chernozhukov, and Fernández-Val (2006).

Assumption A1. *The conditional density $f_{T|X,Z}(t|x, z)$ is bounded and uniformly continuous in t, uniformly for $x \in \mathcal{X}$, $z = 0, 1$. $\mathbb{E}\left[\|X\|^3\right] < \infty$. Let $\vartheta(v):=\mathbb{E}\left[f_{T|X,Z}(S'a(v)|X, Z) SS'\right]$, where*

[12] Under treatment rank invariance, monotonicity holds automatically if treatment quantile changes do not switch signs. This is not true in general under treatment rank similarity. Under rank invariance, if $U_0 = u$, then $U_1 = u$ and vise versa; however, there is no such one-to-one mapping in the counterfactual treatment ranks under treatment rank similarity, and hence monotonic treatment quantile changes do not guarantee individual level monotonicity.

$S{:=}(1, X', Z, ZX')'$, *be positive definite for all* $\upsilon \in \mathcal{V}$ *which is a closed subset of* $(0, 1)$.

Let $e = Y - \mathbb{E}[Y|Z, X, T]$. Let $G = \mathbb{E}\left[\psi^J(X, T, Z)\psi^J(X, T, Z)'\right] = \mathbb{E}\left[\Psi'\Psi/n\right]$ be positive definite for each J. Let $\Omega = \mathbb{E}\left[e^2\psi^J(X, T, Z)\psi^J(X, T, Z)'\right]$ and $\mho = G^{-1}\Omega G^{-1}$.

Let $L^\infty(T)$ denote the set of all bounded measurable functions $g \;\; \mathcal{T} \to \mathcal{R}$ endowed with the sup-norm $\|g\|_\infty = \sup_t |g(t)|$. Let $\|\cdot\|_{\ell^q}$ denote the vector ℓ^q-norm when applied to vectors and the operator norm induced by the vector ℓ^q-norm when applied to matrices. If $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers, then we say $a_n \lesssim b_n$ if $\limsup_{n\to\infty} a_n/b_n < \infty$.

Consider a collection of linear functionals $\{L_\ell \;\; \ell \in \mathcal{L}\}$ with an index set $\mathcal{L}$. For example, for the conditional mean function $m_z(x, t)$, one can let $L_\ell(m_z) = m_z(x, t)$ with $\ell = (x, t) \in \mathcal{L} = \mathcal{X} \times \mathcal{T}$, for $z = 0, 1$. Assumptions A2 and A3 below collect the assumptions in Chen and Christensen (2018).

Assumption A2. *1.* *(i)* (X, T) *have compact rectangular support* $\mathcal{XT} \subset \mathcal{R}^{d_x+1}$ *and the density of* (X, T) *is uniformly bounded away from* 0 *and* ∞ *on* $\mathcal{XT}$.

(ii) For $z = 0, 1$, $m_z \in \mathcal{H} \subset L^\infty(X, T)$. *The sieve space for* (X, T) *is the closed linear span* $\Psi_J = clsp\{\psi_{J1}, ..., \psi_{JJ}\} \subset L^2(X, T)$, *and* $\cup_J \Psi_J$ *is dense in* $(\mathcal{H}, \|\cdot\|_{L^\infty(X,T)})$.

2. *(i)* $\mathbb{E}\left[|e_i|^{2+\delta}\right] < \infty$ *for some* $\delta > 0$.

(ii) $\mathbb{E}\left[|e_i|^3|Z_i = z, X_i = x, T_i = t\right] < \infty$ *and* $\mathbb{E}\left[e_i^2|Z_i = z, X_i = x, T_i = t\right] \in \underline{\sigma}^2, \bar{\sigma}^2$ *for some finite and positive constants* $(\underline{\sigma}^2, \bar{\sigma}^2)$, *uniformly for* $(x, t) \in \mathcal{XT}$, *for* $z = 0, 1$.

3. *(i)* Ψ_J *is Hölder continuous: there exist finite constants* $C \geq 0$, $\tilde{C} > 0$ *such that*
$\|G^{-1/2}\{\psi^J(x, t, z) - \psi^J(\tilde{x}, \tilde{t}, z)\}\|_{\ell^2} \lesssim J^C\|(x, t) - (\tilde{x}, \tilde{t})\|_{\ell^2}^{\tilde{C}}$ *for* $t, \tilde{t} \in \mathcal{T}$, $x, \tilde{x} \in \mathcal{X}$, $z = 0, 1$.

(ii) Let $\zeta{:=}\sup_{x,t,z}\|G^{-1/2}\psi^J(x, t, z)\|_{\ell^2}$ *satisfy* $\zeta^2/\sqrt{n} = O(1)$ *and* $\zeta^{(2+\delta)/\delta}$ $\sqrt{(\log n)/n} = o(1)$.

4. *(i) Let* $\sigma_n^2(L_\ell) = L_\ell(\psi^J)'\mho L_\ell(\psi^J) \nearrow +\infty$ *as* $n \to \infty$ *for each* $\ell \in \mathcal{L}$. *Let* η_n *be a sequence of nonnegative numbers such that* $\eta_n = o(1)$. *Let* $\tilde{m}_z(x, t) = \psi^J(x, t, z)'\tilde{c}$ *where* $\tilde{c} = (\Psi'\Psi)^{-1}\Psi'\big(m_{Z_1}(X_1, T_1), ..., m_{Z_n}(X_n, T_n)\big)'$ *and* $\sup_{\ell\in\mathcal{L}}\sqrt{n}|L_\ell(\tilde{m}_z(x, t)) - L_\ell(m_z(x, t))|/\sigma_n(L_\ell) = O_p(\eta_n)$.

(ii) *Let* $u_n(L_\ell)(X_i,T_i,Z_i)=\psi^J(X_i,T_i,Z_i)'G^{-1}L_\ell(\psi^J)/\sigma_n(L_\ell)$ *be the normalized sieve Riesz representer. Let* $d_n(\ell_1,\ell_2)=\big(\mathbb{E}\big[(u_n(L_{\ell_1})(X_i,T_i,Z_i)-u_n(L_{\ell_2})(X_i,T_i,Z_i))^2\big]\big)^{1/2}$ *be the semimetric on* $\mathcal{L}$. *Let* $N(\mathcal{XT},d_n,\varsigma)$ *be the* ς*-covering number of* $\mathcal{XT}$ *with respect to* d_n. *There is a sequence of finite constant* $c_n\gtrsim 1$ *that could grow to infinity such that* $1+\int_0^\infty\sqrt{\log N(\mathcal{XT},d_n,\varsigma)}d\varsigma=O(c_n)$.

(iii) *Let* $\delta_{m,n}$ *be a sequence of positive constants such that* $\|\hat{m}_z-m_z\|_\infty=O_p(\delta_{m,n})=o_p(1)$. *Define* $\delta_{V,n}:=\big(\zeta^{(2+\delta)/\delta}\sqrt{(\log J)/n}\big)^{\delta/(1+\delta)}+\delta_{m,n}+\zeta\sqrt{(\log J)/n}$. *There is a sequence of constant* $r_n>0$ *decreasing to zero slowly such that (a)* $r_nc_n\lesssim 1$ *and* $\zeta J^2/(r_n^3\sqrt{n})=o(1)$, *(b)* $\zeta\sqrt{(J\log J)/n}+\eta_n+\delta_{V,n}c_n=o(r_n)$.

Assumption A3. *Let* $J\sqrt{(J\log J)/n}=o(1)$. *Let* $B^p_{\infty,\infty}$ *denote the Hölder space of smoothness* $p>0$ *and* $\|\cdot\|_{B^p_{\infty,\infty}}$ *denote its norm. Let* $B_\infty(p,L)=\{m\in B^p_{\infty,\infty}\ \ \|m\|_{B^p_{\infty,\infty}}\le L\}$ *denote a Hölder ball of smoothness* $p>1$ *and radius* $L\in(0,\infty)$. *Let* $m\in B_\infty(p,L)$ *and* Ψ_J *be spanned by a B-spline basis of order* $\gamma>p$ *or a CDV wavelet basis of regularity* $\gamma>p$.

Assumption A3 ensures the uniform consistency of $\partial_t\hat{m}_z(x,t)=\partial\hat{m}_z(x,t)/\partial t$, which is used to account for the Step 1 estimation error.

S.4 Inference for $\pi(x,\upsilon)$

Let the sieve variance estimator for $\hat{\pi}(x,\upsilon)$ be $\hat{\sigma}^2(x,\upsilon)=\Delta\hat{\psi}(x,\upsilon)'\hat{\mho}\Delta\hat{\psi}(x,\upsilon)/\Delta\hat{q}(x,\upsilon)^2$, where $\Delta\hat{\psi}(x,\upsilon)=\psi^J(x,\hat{q}_1(x,\upsilon),1)-\psi^J(x,\hat{q}_0(x,\upsilon),0)$. The $100(1-\alpha)\%$ confidence interval for $\pi(x,\upsilon)$ can be constructed as $\big[\hat{\pi}(x,\upsilon)-z^*_{1-\alpha}\hat{\sigma}(x,\upsilon)/\sqrt{n},\hat{\pi}(x,\upsilon)+z^*_{1-\alpha}\hat{\sigma}(x,\upsilon)/\sqrt{n}\big]$, where the critical value $z^*_{1-\alpha}$ can be $\Phi^{-1}(1-\alpha/2)$ by the asymptotically normal approximation.

Theorem 6. *Let Assumptions A1-A3 hold. Then* $\sqrt{n}(\hat{\pi}(x,\upsilon)-\pi(x,\upsilon))/\hat{\sigma}(x,\upsilon)$ $\xrightarrow{d}\mathcal{N}(0,1)$ *uniformly for* $(x,\upsilon)\in\Upsilon=\{(x,\upsilon)\in\mathcal{X}\times\mathcal{V}\ \ |\Delta q(x,\upsilon)|\ge 0\}$.

For the uniform confidence interval over $(x,\upsilon)\in\Upsilon$, the critical value $z^*_{1-\alpha}$ is simulated from the bootstrap sieve t-statistic $\mathbb{Z}^*_n(x,\upsilon)$ for $(x,\upsilon)\in\Upsilon$: Let $\varpi_1,\ldots,\varpi_n$ be i.i.d. random variables independent of the data with mean zero, unit variance, and finite third moment, e.g., $\mathcal{N}(0,1)$. Let

$$\mathbb{Z}_n^*(x,\upsilon)=\frac{\Delta\hat{\psi}(x,\upsilon)'\hat{G}^{-1}}{\Delta\hat{q}(x,\upsilon)\hat{\sigma}(x,\upsilon)\sqrt{n}}\sum_{i=1}^{n}\psi^J(x,T,Z_i)\hat{e}_i\varpi_i.$$

Calculate $\mathbb{Z}_n^*(x,\upsilon)$ for a large number of independent draws of $\varpi_1,...,\varpi_n$. Then the critical value $z_{1-\alpha}^*$ is the $(1-\alpha)$ quantile of $\sup_{(x,\upsilon)\in\Upsilon}|\mathbb{Z}_n^*(x,\upsilon)|$ over the draws. Theorem 4.1 in Chen and Christensen (2018) implies the result on the consistency of the sieve score bootstrap. $\sup_{s\in\mathcal{R}}\left|\mathbb{P}\Big(\sup_{(x,\upsilon)\in\Upsilon}|\sqrt{n}\big(\hat{\pi}(x,\upsilon)-\pi(x,\upsilon)\big)/\hat{\sigma}(x,\upsilon)|\le s\Big)-\mathbb{P}^*\Big(\sup_{(x,\upsilon)\in\Upsilon}|\mathbb{Z}_n^*(x,\upsilon)|\le s\Big)\right|=o_p(1)$, where $\mathbb{P}^*$ denotes a probability measure conditional on the data $\{Y_i,T_i,X_i,Z_i\}_{i=1}^n$.

S.5 Proofs: Estimation and Inference

The following proofs apply the results of Angrist, Chernozhukov, and Fernańdez-Val (2006) (ACF, henceforth) and Chen and Christensen (2018) (CC, henceforth). To simplify exposition, we collect notations used in the proofs below. We suppress the subscripts i,z and dependence on υ, when there is no confusion.

Notation:

$$\begin{aligned}
\phi_i(\upsilon)&=\vartheta(\upsilon)^{-1}\big(1(T_i\le S_i'a(\upsilon))-\upsilon\big)S_i\\
S_{1i}&=(1,X_i',1,X_i')',\ S_{0i}=(1,X_i',0,\mathbf{0}'_{(d_x\times 1)})',\ \Delta S_i=S_{1i}-S_{0i}\\
\partial_t m_z(X,q_z(X,\upsilon))&=\frac{\partial}{\partial t}m_z(X,t)|_{t=q_z(X,\upsilon)}\\
q_{zi}&=q_z(X_i,\upsilon),\ \hat{q}_{zi}=\hat{q}_z(X_i,\upsilon)\\
\Delta q_i&=\Delta q(X_i,\upsilon)=q_{1i}-q_{0i}=(S_{1i}-S_{0i})'a(\upsilon)=\Delta S_i'a(\upsilon)\\
\Delta\hat{q}_i&=\Delta\hat{q}(X_i,\upsilon)=\hat{q}_{1i}-\hat{q}_{0i}=(S_{1i}-S_{0i})'\hat{a}(\upsilon)=\Delta S_i'\hat{a}(\upsilon)\\
\Delta\psi_i&=\Delta\psi(X_i,\upsilon)=\psi^J(X_i,q_1(X_i,\upsilon),1)-\psi^J(x,q_0(X_i,\upsilon),0)
\end{aligned}$$

$$
\begin{aligned}
\Delta\hat{\psi}_i &= \Delta\hat{\psi}(X_i, \upsilon) = \psi^J(X_i, \hat{q}_1(X_i,\upsilon), 1) - \psi^J(X_i, \hat{q}_0(X_i,\upsilon), 0) \\
\Delta m_i &= \Delta m(X_i, \upsilon) = m_1(X_i, q_1(X_i,\upsilon)) - m_0(X_i, q_0(X_i,\upsilon)) \\
\Delta\hat{m}_i &= \Delta\hat{m}(X_i, \upsilon) = \hat{m}_1(X_i, \hat{q}_1(X_i,\upsilon)) - \hat{m}_0(X_i, \hat{q}_0(X_i,\upsilon)) = \Delta\hat{\psi}_i'\hat{c} \\
\Delta\check{m}_i &= \Delta\check{m}(X_i, \upsilon) = \hat{m}_1(X_i, q_1(X_i,\upsilon)) - \hat{m}_0(X_i, q_0(X_i,\upsilon)) = \Delta\psi_i'\hat{c} \\
\chi_i &= \chi(X_i,\upsilon) = 1(|\Delta q(X_i,\upsilon)| \geq 0|) \\
\chi_i^{\pm} &= \chi^{\pm}(X_i,\upsilon) = 1(\pm\Delta q(X_i,\upsilon) \geq 0|)
\end{aligned}
$$

Lemma 5 is for estimating the sign function.

Lemma 5. *Let Assumption A1 hold. Let $\sqrt{n}l^{-1} = o(1)$. Then*

1.

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{l}\sum_{\upsilon\in V^{(l)}}\Delta m(X_i,\upsilon)\left(\hat{\chi}^+(X_i,\upsilon) - \chi^+(X_i,\upsilon)\right) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^1 \frac{\partial}{\partial\alpha}\mathbb{E}\left[\Delta m(X,\upsilon)1(\Delta S'\alpha \geq 0)\right]'\big|_{\alpha=a(\upsilon)}\phi_i(\upsilon)d\upsilon + o_p(1).
\end{aligned}
$$

2.

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{l}\sum_{\upsilon\in V^{(l)}}\Delta q(X_i,\upsilon)\left(\hat{\chi}^+(X_i,\upsilon) - \chi^+(X_i,\upsilon)\right) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^1 \frac{\partial}{\partial\alpha}\mathbb{E}\left[\Delta q(X,\upsilon)1(\Delta S'\alpha \geq 0)\right]'\big|_{\alpha=a(\upsilon)}\phi_i(\upsilon)d\upsilon + o_p(1).
\end{aligned}
$$

Step 1 is $O_p(n^{-1/2})$, so the estimation error of χ is of first order asymptotically by Lemma 5.

Lemma 6 is for the approximation error from the numerical integration.

Lemma 6. *Let a function $f(x,\upsilon)$ be of bounded variation in $\upsilon \in \mathcal{V}$, uniformly in $x \in \mathcal{X}$. Then*

$$
\sup_{x\in\mathcal{X}}\left|l^{-1}\sum_{\upsilon\in V^{(l)}} f(x,\upsilon)1(\Delta q(x,\upsilon) > 0) - \int_0^1 f(x,\upsilon)1(\Delta q(x,\upsilon) > 0)d\upsilon\right| = O(l^{-1}).
$$

The inference theory for $\pi(\upsilon)$ follows analogously to that of π^{DR}, but without integrating over υ. Therefore we first present the proof of Theorem 4 for π^{DR}.

Proof of Theorem 4: Define A_+ and A_- as $A_\pm = \int_0^1 \int_{\mathcal{X}} \Delta m(x,\upsilon)\chi^{\pm}(x,\upsilon) f_X(x)dxd\upsilon$. So $A = A_+ - A_- = \int_0^1 \int_{\mathcal{X}} \Delta m(x,\upsilon)/\Delta q(x,\upsilon)\big(\Delta q(x,\upsilon)1(\Delta q(x,\upsilon) \geq 0) - \Delta q(x,\upsilon)1(\Delta q(x,\upsilon) \leq 0)\big) f_X(x)dxd\upsilon = \int_0^1 \int_{\mathcal{X}} \pi(x,\upsilon)|\Delta q(x,\upsilon)|1(|\Delta q(x,\upsilon)| \geq 0) f_X(x)dxd\upsilon$.

Define B_+ and B_- as $B_\pm = \int_0^1 \int_{\mathcal{X}} \Delta q(x,\upsilon)\chi^{\pm}(x,\upsilon) f_X(x)dxd\upsilon$. By a similar argument as A, we can show that $B = B_+ - B_-$. Therefore, $\pi^{DR} = A/B$ and $\pi^{DR}_{\pm} = A_\pm/B_\pm$. Linearize $\hat{\pi}^{DR} - \pi^{DR} = (\hat{A} - A)/B - (\hat{B} - B)\pi/B + O_p\Big(|\hat{A} - A||\hat{B} - B|/B^2 + |\hat{B} - B|^2/B^2\Big)$.

The proof focuses on $\hat{A}_+$, the estimator of A_+. The same arguments apply to $\hat{B}_+$, the estimator of B_+. The same arguments apply to $\hat{\pi}^{DR}_{-}$ and hence $\hat{\pi}^{DR}$.

Write $\hat{\pi}^{DR}_{+} = \hat{A}_+/\hat{B}_+$, where

$$\hat{A}_+ = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{l}\sum_{\upsilon \in V^{(l)}} \Delta\hat{m}(X_i,\upsilon)\hat{\chi}^{+}(X_i,\upsilon),$$
$$\hat{B}_+ = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{l}\sum_{\upsilon \in V^{(l)}} \Delta\hat{q}(X_i,\upsilon)\hat{\chi}^{+}(X_i,\upsilon).$$

In the following, we suppress the subscripts of $+$ and superscripts of DR for expositional simplicity. Linearize $\hat{\pi} - \pi = (\hat{A} - A)/B - (\hat{B} - B)\pi/B + O_p\Big(|\hat{A} - A||\hat{B} - B|/B^2 + |\hat{B} - B|^2/B^2\Big)$.

Let $\tilde{A} = n^{-1}\sum_{i=1}^{n} l^{-1}\sum_{\upsilon \in V^{(l)}} \Delta\hat{m}(X_i,\upsilon)\chi(X_i,\upsilon)$ for a known sign function. Decompose $\hat{A} - A = \hat{A} - \tilde{A} + \tilde{A} - A$. The estimation error in $\Delta\hat{m}$.

$$\tilde{A} - A = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{l}\sum_{\upsilon \in V^{(l)}} \big(\Delta\hat{m}(X_i,\upsilon) - \Delta m(X_i,\upsilon)\big)\chi(X_i,\upsilon) \tag{S.4}$$
$$+ \frac{1}{n}\sum_{i=1}^{n}\frac{1}{l}\sum_{\upsilon \in V^{(l)}} \Delta m(X_i,\upsilon)\chi(X_i,\upsilon) - A. \tag{S.5}$$

By Lemma 6 and assuming $\sqrt{n}l^{-1} = o(1)$, (S.5) is $n^{-1}\sum_{i=1}^{n} R_{A3i} + o_p(n^{-1/2})$, where $R_{A3i} = \int_0^1 \Delta m(X_i, v)\chi^+(X_i, v)dv - A_+$.

We focus on (S.4) next. Decompose $\Delta\hat{m}_i - \Delta m_i = \left(\Delta\hat{m}_i - \Delta\check{m}_i\right) + \left(\Delta\check{m}_i - \Delta m_i\right)$. The first part is for Step 1 estimation error, and the second part is for Step 2 estimation error.

Step 1 Theorem 3 in ACF shows that $\hat{a}(v) - a(v) = n^{-1}\sum_{i=1}^{n}\phi_i(v) + o_p(n^{-1/2})$ uniformly over $v \in \mathcal{V}$ and converges in distribution to a zero mean Gaussian process indexed by v. Decompose

$$
\begin{aligned}
&\Delta\hat{m}_i - \Delta\check{m}_i \\
&= m_1(X_i, \hat{q}_{1i}) - m_1(X_i, q_{1i}) - (m_0(X_i, \hat{q}_{0i}) - m_0(X_i, q_{0i})) + so1 \\
&= \partial_t m_1(X_i, q_{1i})(\hat{q}_{1i} - q_{1i}) - \partial_t m_0(X_i, q_{0i})(\hat{q}_{0i} - q_{0i}) + so1 + so2 \\
&= \partial_t m_1(X_i, q_{1i})S_{1i}(\hat{a}(v) - a(v)) - \partial_t m_0(X_i, q_{0i})S_{0i}(\hat{a}(v) - a(v)) + so1 + so2,
\end{aligned}
$$

where (We suppress the subscript i for simplicity.)

$$
\begin{aligned}
so1 &= \hat{m}_1(\hat{q}_1) - m_1(\hat{q}_1) - \left(\hat{m}_0(\hat{q}_0) - m_0(\hat{q}_0)\right) - \left(\hat{m}_1(q_1) - m_1(q_1)\right) \\
&\quad + \left(\hat{m}_0(q_0) - m_0(q_0)\right) \\
&= O_p\left(\|\partial_t\hat{m}_z - \partial_t m_z\|_\infty \|\hat{q}_z - q_z\|_\infty\right), \\
so2 &= O_p\left(\partial_t^2 m_1\left(\hat{q}_1 - q_1\right)^2 + \partial_t^2 m_0\left(\hat{q}_0 - q_0\right)^2\right) = O_p(\|\hat{q}_z - q_z\|_\infty^2),
\end{aligned}
$$

as $\partial_t^2 m_z$ is uniformly bounded by Assumption A3. ACF and Corollary 3.1(ii) in CC implies that $so1 + so2 = O_p(\|\hat{q}_z - q_z\|_\infty \|\partial_t\hat{m}_z - \partial_t m_z\|_\infty + \|\hat{q}_z - q_z\|_\infty^2) = O_p(n^{-1/2}(J^{-(p-1)} + J\sqrt{(J\log J)/n}) + n^{-1}) = o_p(n^{-1/2})$ uniformly over $v \in \mathcal{V}$, by assuming $J\sqrt{(J\log J)/n} = o(1)$ and $p > 1$.

Then

$$
\begin{aligned}
&\sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{l}\sum_{v\in V^{(l)}}(\Delta\hat{m}_i-\Delta\check{m}_i)\chi_i\\
&=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{l}\sum_{v\in V^{(l)}}(\partial_t m_1(X_i,q_{1i})S_{1i}'\chi_i\sqrt{n}(\hat{a}(v)-a(v))\\
&-\partial_t m_0(X_i,q_{0i})S_{0i}'\chi_i\sqrt{n}(\hat{a}(v)-a(v))+o_p(1)\\
&=\frac{1}{l}\sum_{v\in V^{(l)}}\mathbb{E}\left[(\partial_t m_1(X_i,q_{1i})S_{1i}-\partial_t m_0(X_i,q_{0i})S_{0i})\,\chi_i\right]'\sqrt{n}(\hat{a}(v)-a(v))+o_p(1)\\
&=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{l}\sum_{v\in V^{(l)}}\mathbb{E}\left[(\partial_t m_1(X_i,q_{1i})S_{1i}-\partial_t m_0(X_i,q_{0i})S_{0i})\,\chi_i\right]'\phi_j(v)+o_p(1)\\
&=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\int_0^1\mathbb{E}\left[(\partial_t m_1(X_i,q_{1i})S_{1i}-\partial_t m_0(X_i,q_{0i})S_{0i})\,\chi_i\right]'\phi_j(v)dv+o_p(1),
\end{aligned}
$$

where the third equality is by ACF, and the last equality is by Lemma 6 and $\sqrt{n}l^{-1}=o(1)$.

For the second equality, let $\mathcal{F}=\left\{1(\Delta S_i'a>0),\,a\in\mathcal{B}\right\}$ that is a VC subgraph class and hence a bounded Donsker class. Then $\mathcal{F}(\partial_t m_1(X_i,S_{1i}'a)S_{1i}-\partial_t m_0(X_i,S_{0i}'a)S_{0i})$ is also bounded Donsker with a square-integrable envelop $2\sup_{z,x,t}|\partial_t m_z(x,t)|\max_{j\in\{1,2,\ldots,d_x\}}|X_j|$ by Theorem 2.10.6 in Van der Vaart and Wellner (1996). So $n^{-1}\sum_{i=1}^{n}(\partial_t m_1(X_i,q_{1i})S_{1i}-\partial_t m_0(X_i,q_{0i})S_{0i})\chi_i=\mathbb{E}(\partial_t m_1(X_i,q_{1i})S_{1i}-\partial_t m_0(X_i,q_{0i})S_{0i})\chi_i+o_p(n^{-1/2})$ uniformly in $v\in\mathcal{V}$.

Step 2 We show the stochastic equicontinuity, $n^{-1}\sum_{i=1}^{n}(\Delta\check{m}_i-\Delta m_i)\chi_i=\mathbb{E}\left[(\Delta\check{m}_i-\Delta m_i)\chi_i\right]+so3$, where $so3=o_p(n^{-1/2})$ uniformly in $v\in\mathcal{V}$.

Let $\Delta\tilde{m}_i=\Delta\psi_i'\tilde{c}$ and $\Delta\check{m}_i=\Delta\psi_i'\hat{c}$. Then decompose $so3=so31+so32$ to the "standard deviation" term $so31$ and the "bias" term $so32$,

$$
\begin{aligned}
so3=&\frac{1}{n}\sum_{i=1}^{n}\chi_i\left(\Delta\check{m}_i-\Delta\tilde{m}_i\right)-\int_{\mathcal{X}}\chi_i\left(\Delta\check{m}_i-\Delta\tilde{m}_i\right)F_X(dX_i) &\text{(so31)}\\
&+\frac{1}{n}\sum_{i=1}^{n}\chi_i\left(\Delta\tilde{m}_i-\Delta m_i\right)-\int_{\mathcal{X}}\chi_i\left(\Delta\tilde{m}_i-\Delta m_i\right)F_X(dX_i). &\text{(so32)}
\end{aligned}
$$

Let $\sqrt{n}so31=Q_J'(\hat{c}-\tilde{c})$, where $Q_J=\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n}\chi_i\Delta\psi_i-\int_{\mathcal{X}}\chi_i\Delta\psi_iF_X(dX_i))$. By

$var(Q_J) = \mathbb{E}\chi_i \Delta\psi_i \Delta\psi_i'$ and the Jensen's inequality, $\mathbb{E}\|Q_J\| \leq O(\sqrt{\mathbb{E}\|\Delta\psi_i\|^2}) = O(\zeta)$. As given in the proof of Lemma 3.1 in CC, $\|\hat{c} - \tilde{c}\|_{\ell^\infty} = O_p(\sqrt{\log J/(n\lambda_{min}(G))})$, where the minimum eigenvalue $\lambda_{min}(G) > 0$.[13] Then $\mathbb{E}|so31| = O(n^{-1/2}\zeta\sqrt{\log J/(n\lambda_{min}(G))})$ by the Cauchy-Schwartz inequality. The Markov's inequality implies $so31 = O_p(n^{-1}\zeta\sqrt{\log J/\lambda_{min}(G)}) = o_p(n^{-1/2})$ implied by Assumption A2.5.

$var(\sqrt{n}so32) = O(\mathbb{E}\left[\chi_i\,(\Delta\tilde{m}_i - \Delta m_i)^2\right]) = O(\|m - \Pi_J m\|_\infty^2)$, where $\Pi_J m = \arg\min_{h\in\Psi_J}\|m-h\|_{L^2(X,T,Z)}$, by Theorem 3.1 (i) in CC. The Markov's inequality yields $so32 = O_p(n^{-1/2}\|m - \Pi_J m\|_\infty) = O_p(n^{-1/2}J^{-p}) = o_p(1)$ by the results in the proof of Corollary 3.1 in CC.

By Lemma 6 and assuming $\sqrt{n}l^{-1} = o(1)$, $l^{-1}\sum_{v\in V^{(l)}}\mathbb{E}\left[(\Delta\check{m}_i - \Delta m_i)\chi_i\right] = \int_0^1 \mathbb{E}\left[\Delta\check{m}_i\chi_i\right]dv - A + o_p(n^{-1/2})$.

Note that A is based on a linear functional of m, $L(m) = \int_0^1\int_{\mathcal{X}} m_z(x, q_z(x,v))1(\Delta q(x,v) > 0)F_X(dx)dv$. So we use the results on linear functionals of a sieve estimator in CC. Let $\sigma_{A2n}^2 = \mathbb{E}\left[R_{A2i}^2\right]$, where $R_{A2i} = \mathcal{D}^{+\prime}G^{-1}\psi^J(X_i, T_i, Z_i)e_i$ and $\mathcal{D}^+ = \int_0^1\mathbb{E}\left[\Delta\psi^J(X,v)\chi^+(X,v)\right]dv$, with a consistent estimator $\hat{\sigma}_{A2}^2$. Lemma 4.1 in CC provides

$$\left|\frac{\sqrt{n}}{\hat{\sigma}_{A2}}\left(\int_0^1 \mathbb{E}\left[\Delta\check{m}_i\chi_i\right]dv - A\right) - \frac{1}{\sigma_{A2n}\sqrt{n}}\sum_{i=1}^n R_{A2i}\right| = o_p(1).$$

The estimation error from the sign function $\hat{A} - \tilde{A} = n^{-1}\sum_{i=1}^n l^{-1}\sum_{v\in V^{(l)}}\Delta m(X_i,v)\big(\hat{\chi}(X_i,v) - \chi(X_i,v)\big) + o_p(1/\sqrt{n})$ by $n^{-1}\sum_{i=1}^n l^{-1}\sum_{v\in V^{(l)}}\big(\Delta\hat{m}(X_i,v) - \Delta m(X_i,v)\big)\big(\hat{\chi}(X_i,v) - \chi(X_i,v)\big) = O_p\Big(\|\Delta\hat{m} - \Delta m\|_\infty\|\hat{q}_z - q_z\|_\infty\Big) = o_p(n^{-1/2})$. Together with Lemma 5(i), $\left|\sqrt{n}(\hat{A} - A) - n^{-1/2}\sum_{i=1}^n R_{Ai}\right| = o_p(1)$, where $R_{Ai} = R_{A1i} + R_{A2i} +$

[13] By Lemma A.1 in CC, $s_{JK}^{-1} \asymp \pi_J = 1$ for the exogenous case.

R_{A3i} with

$$R_{A1i} = \int_0^1 \Big(\mathbb{E}\left[(\partial_t m_1(X, q_1) S_1 - \partial_t m_0(X, q_0) S_0)\, \chi^+(X, v) \right] + \frac{\partial}{\partial \alpha} \mathbb{E}\left[\Delta m(X, v) 1(\Delta S' \alpha \geq 0) \right] \big|_{\alpha = a(v)} \Big)' \phi_i(v) dv,$$

By the similar arguments as for A in (S.4) and (S.5),

$$\tilde{B} - B = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{l} \sum_{v \in V^{(l)}} \left(\Delta \hat{q}(X_i, v) - \Delta q(X_i, v) \right) \chi(X_i, v) \tag{S.6}$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{l} \sum_{v \in V^{(l)}} \Delta q(X_i, v) \chi(X_i, v) - B. \tag{S.7}$$

By Lemma 6, (S.7) is $n^{-1} \sum_{i=1}^{n} \int_0^1 \Delta q(X_i, v) \chi(X_i, v) dv - B + o_p(n^{-1/2})$. (S.6) is

$$\begin{aligned}
&\frac{1}{n} \sum_{i=1}^{n} \frac{1}{l} \sum_{v \in V^{(l)}} \Delta S_i' \left(\hat{a}(v) - a(v) \right) \chi_i \\
&= \frac{1}{n} \sum_{j=1}^{n} \frac{1}{l} \sum_{v \in V^{(l)}} \frac{1}{n} \sum_{i=1}^{n} \chi_i \Delta S_i' \phi_j(v) + o_p(n^{-1/2}) \\
&= \frac{1}{n} \sum_{j=1}^{n} \frac{1}{l} \sum_{v \in V^{(l)}} \mathbb{E}\left[\chi_i \Delta S_i \right]' \phi_j(v) + o_p(n^{-1/2})
\end{aligned}$$

$$= \frac{1}{n} \sum_{j=1}^{n} \int_0^1 \mathbb{E}\left[\Delta S_i' \chi_i \right] \phi_j(v) dv + o_p(n^{-1/2}),$$

where the first equality by ACF, and the third equality by Lemma 6. For the second equality, let $\mathcal{F} = \left\{ 1(\Delta S_i' a > 0), a \in \mathcal{B} \right\}$ that is a VC subgraph class and hence a bounded Donsker class. Then $\mathcal{F} \Delta S$ is Donsker with a square-integrable envelop $\max_{j \in \{1,2,\ldots,d_x\}} |X_j|$ by Theorem 2.10.6 in Van der Vaart and Wellner (1996). So $n^{-1} \sum_{i=1}^{n} \chi_i \Delta S_i - \mathbb{E}\left[\chi_i \Delta S_i \right] = o_p(1)$ uniformly over $v \in \mathcal{V}$

Together with Lemma 5(ii), we obtain $\left| \sqrt{n}(\hat{B} - B) - n^{-1/2} \sum_{i=1}^{n} R_{Bi} \right| = o_p(1)$, where $R_{Bi} =$

$R_{B1i} + R_{B3i}$ with

$$R_{B1i} = \int_0^1 \left(\mathbb{E}\left[\Delta S' \chi^+(X,\upsilon)\right] + \frac{\partial}{\partial \alpha}\mathbb{E}\left[\Delta q(X,\upsilon) 1(\Delta S'\alpha \geq 0)\right]' \big|_{\alpha = a(\upsilon)} \right) \phi_i(\upsilon) d\upsilon$$
$$R_{B3i} = \int_0^1 \Delta q(X_i, \upsilon)\chi^+(X_i,\upsilon) d\upsilon - B.$$

By a linearization for $\hat{\pi}_+^{DR}$, $\hat{\pi}_+^{DR} - \pi_+^{DR} = \frac{\hat{A}_+}{\hat{B}_+} - \frac{A_+}{B_+} = \frac{\hat{A}_+ - A_+}{B_+} - \frac{\pi_+^{DR}}{B_+}(\hat{B}_+ - B_+) + o_p(n^{-1/2})$. Therefore, we define $R_i^+ = R_{Ai} - \pi_+^{DR} R_{Bi} = R_{1i}^+ + R_{2i}^+ + R_{3i}^+$, where $R_{1i}^+ = R_{A1i} - \pi_+^{DR} R_{B1i}$, $R_{2i}^+ = R_{A2i}$, and $R_{3i}^+ = R_{A3i} - \pi_+^{DR} R_{B3i}$. That is,

$$R_{1i}^+ = \int_0^1 \Bigg(\mathbb{E}\left[\left(\partial_t m_1(X, q_1(X,\upsilon)) S_1 - \partial_t m_0(X, q_0(X,\upsilon)) S_0 - \pi_+^{DR} \Delta S\right) \chi^+(X,\upsilon)\right]$$
$$+ \frac{\partial}{\partial \alpha}\mathbb{E}\left[\left(\Delta m(X,\upsilon) - \pi_+^{DR}\Delta q(X,\upsilon)\right) 1(\Delta S'\alpha \geq 0)\right] \big|_{\alpha = a(\upsilon)} \Bigg)' \phi_i(\upsilon) d\upsilon,$$
$$R_{2i}^+ = \mathcal{D}^{+'} G^{-1} \psi^J(X_i, T_i, Z_i) e_i, \text{ with } \mathcal{D}^+ = \int_0^1 \mathbb{E}\left[\Delta \psi^J(X,\upsilon)\chi^+(X,\upsilon)\right] d\upsilon,$$
$$R_{3i}^+ = \int_0^1 \left(\Delta m(X_i,\upsilon) - \pi_+^{DR}\Delta q(X_i,\upsilon)\right) \chi^+(X_i,\upsilon) d\upsilon.$$

Then we obtain $\hat{\pi}_+^{DR} - \pi_+^{DR} = n^{-1}\sum_{i=1}^n \left(R_{Ai} - \pi_+^{DR} R_{Bi}\right)/B_+ + o_p(n^{-1/2}) = n^{-1}\sum_{i=1}^n R_i^+/B_+ + o_p(n^{-1/2})$.

Asymptotic normality We suppress the subscripts of $+$ and superscripts of DR for expositional simplicity. Because R_{2i} depends on (Y_i, T_i, X_i), R_{1i} depends on (T_i, X_i), and R_{3i} depends on X_i, the law of iterated expectations yields $\sigma_n^2 = \left(\mathbb{E}\left[R_{1i}^2\right] + \mathbb{E}\left[R_{2i}^2\right] + \mathbb{E}\left[R_{3i}^2\right]\right)/B^2 = (\sigma_1^2 + \sigma_{2n}^2 + \sigma_3^2)/B^2$.

We will show the Bahadur representation that

$$\left| \frac{\sqrt{n}(\hat{\pi} - \pi)}{\hat{\sigma}} - \frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{R_i}{B\sigma_n} \right|$$
$$\leq \left| \frac{\sqrt{n}(\hat{\pi} - \pi)}{\sigma_n} - \frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{R_i}{B\sigma_n} \right| + \left| \frac{\sqrt{n}(\hat{\pi} - \pi)}{\sigma_n}\left(\frac{\sigma_n}{\hat{\sigma}} - 1\right) \right| = o_p(1)$$

by (i) $n^{-1/2}\sum_{i=1}^{n} R_i/(B\sigma_n) \xrightarrow{d} \mathcal{N}(0,1)$, and (ii) $|\sigma_n/\hat{\sigma} - 1| = o_p(1)$, as shown below.

(i) Asymptotic normality will follow from the Lyapunov central limit theorem with the third absolute moment, $n^{-1/2}\mathbb{E}|R_i|^3/(B\sigma_n)^3 \to 0$, since $\{R_i\}_{i=1}^n$ are independent across i, with mean zero and variance 1. By the assumed conditions, it is straightforward to show that $n^{-1/2}\mathbb{E}|R_{1i}|^3/(B\sigma_1)^3 \to 0$. We show below that $n^{-1/2}\mathbb{E}|R_{2i}|^3/(B\sigma_{2n})^3 \to 0$. Then it implies that all the cross-product terms $n^{-1/2}\mathbb{E}|R_{1i}R_{2i}R_{3i}|/(B\sigma_n)^3 \to 0$ and $n^{-1/2}\mathbb{E}|R_{ji}^2 R_{ki}|/(B\sigma_n)^3 \to 0$ for $j,k = 1,2,3$, $j \neq k$.

Denote as $\psi_i = \psi^J(X_i, T_i, Z_i)$. By Assumption A2.2(ii),

$$\begin{aligned}\sigma_{2n}^2 &= \mathbb{E}\left[R_{2i}^2\right]/B^2 = \mathbb{E}\left[(\mathcal{D}'G^{-1}\psi_i)^2 e_i^2\right]/B^2 \\ &\geq \mathbb{E}\left[(\mathcal{D}'G^{-1}\psi_i)^2\right]\underline{\sigma}^2/B^2 = \mathcal{D}'G^{-1}\mathcal{D}\underline{\sigma}^2/B^2. \end{aligned} \tag{S.8}$$

By the Schwarz inequality, (S.8), and Assumption A2.3(ii),

$$\frac{(\mathcal{D}'G^{-1}\psi_i)^2}{\sigma_{2n}^2} \leq \frac{(\mathcal{D}'G^{-1}\mathcal{D}')(\psi_i'G^{-1}\psi_i)}{\sigma_{2n}^2} \leq \frac{\zeta^2}{\underline{\sigma}^2}. \tag{S.9}$$

Then by (S.8), (S.9), and Assumption A2.2(iii),

$$\begin{aligned}\frac{1}{\sqrt{n}}\mathbb{E}\left[\frac{|R_{2i}|^3}{B^3\sigma_{2n}^3}\right] &= \frac{1}{\sqrt{n}}\mathbb{E}\left[\frac{|\mathcal{D}'G^{-1}\psi_i e_i|^3}{B^3\sigma_{2n}^3}\right] \\ &= \frac{1}{\sqrt{n}}\mathbb{E}\left[\frac{(\mathcal{D}'G^{-1}\psi_i)^2}{B^3\sigma_{2n}^2}\frac{|\mathcal{D}'G^{-1}\psi_i|}{\sigma_{2n}}\mathbb{E}\left[|e_i|^3|X_i, T_i, Z_i\right]\right] \\ &\leq \frac{\zeta}{\sqrt{n}B^3\underline{\sigma}^3}\sup_{x,t,z}\mathbb{E}\left[|e_i|^3|X_i = x, T_i = t, Z_i = z\right] = O\left(\frac{\zeta}{\sqrt{n}}\right) = o(1).\end{aligned}$$

(ii) It is straightforward that $\hat{\sigma}_1^2 = n^{-1}\sum_{i=1}^n \hat{R}_{1i}^2/\hat{B}^2 \xrightarrow{p} \sigma_1^2 = \mathbb{E}\left[R_{1i}^2\right]/B^2$ and $\hat{\sigma}_3^2 \xrightarrow{p} \sigma_3^2$. The same arguments in Lemma G.4 in CC give $|\sigma_{2n}/\hat{\sigma}_2 - 1| = O_p(\delta_{V,n}) = o_p(1)$. So $|\sigma_n/\hat{\sigma} - 1| = o_p(1)$.

By (i) that $n^{-1/2}\sum_{i=1}^n R_i/(B\sigma_n) = O_p(1)$ and (ii), the second term $\left|\frac{\sqrt{n}(\hat{\pi}-\pi)}{\hat{\sigma}}\left(\frac{\hat{\sigma}}{\sigma_n} - 1\right)\right| =$

$O_p(1)o_p(1) = o_p(1)$. We then obtain the Bahadur representation. The asymptotic normality follows from the result (i).

Therefore, we obtain that when $B_+ > 0$, $\sqrt{n}\big(\hat{\pi}_+^{DR} - \pi_+^{DR}\big)/\hat{\sigma}_{n+} = n^{-1/2}\sum_{i=1}^n R_i^+ /(B_+\sigma_{n+}) + o_p(1) \xrightarrow{d} \mathcal{N}(0,1)$, where $\hat{\sigma}_+^2$ is a consistent estimator of $\sigma_{n+}^2 = \mathbb{E}\left[R_i^{+2}\right]/B_+^2$.

For π_-^{DR}, define

$$R_{1i}^- = \int_0^1 \Bigg(\mathbb{E}\left[\left(\partial_t m_1(X, q_1(X,v))S_1 - \partial_t m_0(X, q_0(X,v))S_0 - \pi_-^{DR}\Delta S\right)\chi^-(X,v)\right] \\ + \frac{\partial}{\partial\alpha}\mathbb{E}\left[\left(\Delta m(X,v) - \pi_+^{DR}\Delta q(X,v)\right)1(\Delta S'\alpha \le 0)\right]\big|_{\alpha=a(v)}\Bigg)'\phi_i(v)dv.$$

Define R_i^- as R_i^+ by replacing $+$ with $-$ in all the components in R_i^+. By the same arguments for π_+^{DR}, we obtain that when $B_- > 0$, $\sqrt{n}\big(\hat{\pi}_-^{DR} - \pi_-^{DR}\big)/\hat{\sigma}_- = n^{-1/2}\sum_{i=1}^n R_i^-/(B_-\sigma_{n-}) + o_p(1) \xrightarrow{d} \mathcal{N}(0,1)$, where $\hat{\sigma}_-^2$ is a consistent estimator of $\sigma_{n-}^2 = \mathbb{E}\left[R_i^{-2}\right]/B_-^2$, such that $|\sigma_{n-}/\hat{\sigma}_- - 1| = o_p(1)$.

For π^{DR}, the same linearization yields $\hat{\pi}^{DR} - \pi^{DR} = (\hat{A}-A)/B - (\hat{B}-B)\pi^{DR}/B + O_p\big(|\hat{A}-A||\hat{B}-B|/B^2 + |\hat{B}-B|^2/B^2\big)$. Let $R_i = R_i^+ - R_i^- = R_{1i} + R_{2i} + R_{3i}$, where $R_{li} = R_{li}^+ - R_{li}^-$ for $l = 1,2,3$ by replacing π_+^{DR} and π_-^{DR} with π^{DR}. Specifically, let $\mathrm{sgn}(x,v) = 1(\Delta q(x,v) \ge$

$0) - 1(\Delta q(x, v) \leq 0)$,

$$
\begin{aligned}
R_{1i} &= \int_0^1 \Big(\mathbb{E}\big[(\partial_t m_1(X, q_1(X, v))S_1 - \partial_t m_0(X, q_0(X, v))S_0 - \pi^{DR}\Delta S)\mathrm{sgn}(X, v)\big] \\
&\quad + \frac{\partial}{\partial \alpha}\mathbb{E}\big(\Delta m(X, v) - \pi^{DR}\Delta q(X, v)\big)\,(1(\Delta S'\alpha \geq 0) \\
&\quad - 1(\Delta S'\alpha \leq 0))\big|_{\alpha=a(v)}\Big)'\phi_i(v)dv, \\
&\quad \text{with } \phi_i(v) = \vartheta(v)^{-1}\big(1(T_i \leq S_i'a(v)) - v\big)S_i, \\
&\qquad S_{1i} = (1, X_i', 1, X_i')', S_{0i} = (1, X_i', 0, \mathbf{0}'_{(d_x\times 1)})', \Delta S_i = S_{1i} - S_{0i}, \\
R_{2i} &= \mathcal{D}'G^{-1}\psi^J(X_i, T_i, Z_i)e_i, \\
&\quad \text{with } \mathcal{D} = \int_0^1 \mathbb{E}\left[(\psi^J(X, q_1(X, v), 1) - \psi^J(X, q_0(X, v), 0))\mathrm{sgn}(X, v)\right]dv. \\
R_{3i} &= \int_0^1 \big(\Delta m(X_i, v) - \pi^{DR}\Delta q(X_i, v)\big)\,\mathrm{sgn}(X_i, v)dv, \\
B &= \int_0^1\int_{\mathcal{X}} |\Delta q(x, v)|1(|\Delta q(x, v)| \geq 0)\,f(x)dxdv.
\end{aligned} \tag{S.10}
$$

Proof of Theorem 3: The proof follows exactly the same arguments in the proof of Theorem 4 and Lemma 5 by removing all "$\int_0^1 \cdots dv$" and "$l^{-1}\sum_{v\in V^{(l)}}$". We can derive the influence function of $\hat{\pi}(v)$ to be $R_i(v)/B(v)$ defined as the influence function of $\hat{\pi}^{DR}$ given in (S.10) by removing all $\int_0^1 \cdots dv$. Specifically, as π^{DR}, define $\pi_+(v)$ over units experiencing positive changes for $v \in \mathcal{V}_{+0} = \{v \in \mathcal{V} \quad P(\Delta q(X, v) \geq 0) > 0\}$. Define $B_+(v) = \int_{\mathcal{X}} \Delta q(x, v)\chi^+(x, v)\,f(x)dx$, so $B_+ = \int_0^1 B_+(v)dv$. The influence function of $\hat{\pi}_+(v)$ is $R_i^+(v)/B_+(v) = (R_{1i}^+(v) + R_{2i}^+(v) +$

$R^+_{3i}(v))/B_+(v)$, where

$$R^+_{1i}(v) = \Bigg(\frac{\partial}{\partial\alpha}\mathbb{E}\left[(\Delta m(X,v) - \pi_+(v)\Delta q(X,v))\,1(\Delta S'\alpha \geq 0)\right]\big|_{\alpha=a(v)}$$
$$+ \mathbb{E}\Big[(\partial_t m_1(X, q_1(X,v))S_1 - \partial_t m_0(X, q_0(X,v))S_0$$
$$- \pi_+(v)\Delta S)\chi^+(X,v)\Big]\Bigg)'\phi_i(v),$$
$$R^+_{2i}(v) = \mathcal{D}^{+'}(v)G^{-1}\psi^J(X_i,T_i,Z_i)e_i, \text{ with } \mathcal{D}^+(v) = \mathbb{E}\left[\Delta\psi^J(X,v)\chi^+(X,v)\right],$$
$$R^+_{3i}(v) = (\Delta m(X_i,v) - \pi_+(v)\Delta q(X_i,v))\,\chi^+(X_i,v). \tag{S.11}$$

Similarly consider $\pi_-(v)$ over units experiencing negative changes for $v \in \mathcal{V}_0 = \{v \in \mathcal{V}$ $P(-\Delta q(X,v) \geq 0) > 0\}$. Let $B(v) = B_+(v) - B_-(v)$, where $B_-(v) = \int_{\mathcal{X}} \Delta q(x,v)\chi^-(x,v)f(x)dx$. Let $R_i(v) = R_i^+(v) - R_i^-(v)$, and the influence function of $\hat{\pi}(v)$ is $R_i(v)/B(v)$.

Define $\sigma^2(v) = \mathbb{E}\left[R_i(v)^2\right]/B(v)^2$. The unknown elements are estimated following the same procedure as $\hat{\pi}^{DR}$ by removing "$l^{-1}\sum_{v\in V^{(l)}}$." For example, $\hat{\mathcal{D}}^+(v) = n^{-1}\sum_{i=1}^n \Delta\hat{\psi}_i\hat{\chi}^+(X_i,v)$.

Proof of Theorem 6: We first show that the estimation error of $\hat{q}_z(x,v)$ in Step 1 is of smaller order than the estimation error in Step 2, i.e., the first-order asymptotic distribution of $\hat{\pi}(x,v)$ is as if $q_z(x,v)$ was known. Under Assumption A1, Theorem 3 in ACF implies that $\sup_{(x,v)\in\mathcal{X}\times\mathcal{V}}|\hat{q}_z(x,v) - q_z(x,v)| = O_p(n^{-1/2})$. The Step 2 series least squares estimator converges at a nonparametric rate shower than $\sqrt{n}$. Therefore the first-order asymptotic distribution of $\hat{\pi}(x,v)$ is dominated by Step 2 $\Delta\check{m}(x,v)$.

Step 1 When T_{zi} is observed, i.e., there is no Step 1 estimation error, define $\check{\pi}(x,v) = \Delta\check{m}(x,v)/\Delta q(x,v)$. Decompose $\hat{\pi}(x,v) - \check{\pi}(x,v) = \frac{\Delta\hat{m}}{\Delta\hat{q}} - \frac{\Delta\check{m}}{\Delta q} = \left(\frac{\Delta\hat{m}}{\Delta\hat{q}} - \frac{\Delta\check{m}}{\Delta\hat{q}}\right) + \left(\frac{\Delta\check{m}}{\Delta\hat{q}} - \frac{\Delta\check{m}}{\Delta q}\right)$. The second part is for Step 1 in the denominator: $\frac{\Delta\check{m}}{\Delta\hat{q}} - \frac{\Delta\check{m}}{\Delta q} = \frac{\Delta m}{\Delta q^2}(\Delta q - \Delta\hat{q}) + so1$. The first part is for Step 1 in the argument in the numerator,

$$\begin{aligned}
&\frac{\Delta \hat{m}}{\Delta \hat{q}} - \frac{\Delta \check{m}}{\Delta \hat{q}} \\
&= \frac{1}{\Delta q}(\Delta \hat{m} - \Delta \check{m}) + so2 \\
&= \frac{1}{\Delta q}(m_1(x, \hat{q}_1) - m_1(x, q_1) - (m_0(x, \hat{q}_0) - m_0(x, q_0))) + so2 + so3 \\
&= \frac{1}{\Delta q}(\partial_t m_1(x, q_1)(\hat{q}_1 - q_1) - \partial_t m_0(x, q_0)(\hat{q}_0 - q_0)) + so2 + so3 + so4,
\end{aligned}$$

where

$$\begin{aligned}
so1 &= \frac{\Delta \check{m}}{\Delta \hat{q} \Delta q}(\Delta q - \Delta \hat{q}) - \frac{\Delta m}{\Delta q^2}(\Delta q - \Delta \hat{q}) = (\Delta q - \Delta \hat{q})\frac{1}{\Delta q}\left(\frac{\Delta \check{m}}{\Delta \hat{q}} - \frac{\Delta m}{\Delta q}\right), \\
so2 &= \Delta \hat{m}\left(\frac{1}{\Delta \hat{q}} - \frac{1}{\Delta q}\right) + \Delta \check{m}\left(\frac{1}{\Delta q} - \frac{1}{\Delta \hat{q}}\right) = \left(\Delta \hat{m} - \Delta \check{m}\right)\left(\frac{1}{\Delta \hat{q}} - \frac{1}{\Delta q}\right), \\
so3 &= \frac{1}{\Delta q}\Big\{\hat{m}_1(x, \hat{q}_1) - m_1(x, \hat{q}_1) - \big(\hat{m}_0(x, \hat{q}_0) - m_0(x, \hat{q}_0)\big) - \big(\hat{m}_1(x, q_1) \\
&\quad - m_1(x, q_1)\big) + \big(\hat{m}_0(x, q_0) - m_0(x, q_0)\big)\Big\} \\
&= O_p\left((\partial_t \hat{m}_1(x, q_1) - \partial_t m_1(x, q_1))(\hat{q}_1 - q_1)\right), \\
so4 &= O_p\left(\partial_t^2 m_1\big(\hat{q}_1 - q_1\big)^2 + \partial_t^2 m_0\big(\hat{q}_0 - q_0\big)^2\right) = O_p(\|\hat{q}_z - q_z\|_\infty^2).
\end{aligned}$$

Thus $so1 + so2 + so3 + so4 = O_p(\|\hat{T} - T\|_\infty^2 + \|\hat{T} - T\|_\infty \|\partial_t \check{m} - \partial_t m\|_\infty) = O_p(n^{-1} + n^{-1/2}(J^{-(p-1)} + J\sqrt{(J4\log J)/n})) = o_p(n^{-1/2})$ uniformly over $(x, \upsilon) \in \Upsilon$, by Corollary 3.1(ii) in CC and assuming $J\sqrt{(J\log J)/n}=o(1)$ and $p>1$. Therefore,

$$\begin{aligned}
&\sqrt{n}\big(\hat{\pi}(x,\upsilon)-\check{\pi}(x,\upsilon)\big)\\
&=\sqrt{n}\left\{\frac{\Delta m}{\Delta q^2}(\Delta q-\Delta\hat{q})+\frac{1}{\Delta q}(\partial_t m_1(x,q_1)(\hat{q}_1-q_1)-\partial_t m_0(x,q_0)(\hat{q}_0-q_0))\right\}\\
&\quad+o_p(1)\\
&=\left\{-\frac{\pi(x,\upsilon)}{\Delta q}(S_1-S_0)+\frac{1}{\Delta q}(\partial_t m_1(x,q_1)S_1-\partial_t m_0(x,q_0)S_0)\right\}'\sqrt{n}(\hat{a}(\upsilon)-a(\upsilon))\\
&\quad+o_p(1)\\
&=\left\{-\frac{\pi(x,\upsilon)}{\Delta q}\Delta S+\frac{1}{\Delta q}(\partial_t m_1(x,q_1)S_1-\partial_t m_0(x,q_0)S_0)\right\}'\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\phi_j(\upsilon)+o_p(1) \qquad \text{(S.12)}
\end{aligned}$$

by Theorem 3 in ACF and $\|\hat{\pi}-\check{\pi}\|_\infty=O_p(n^{-1/2})$.

Step 2 Define $\mathcal{Z}_n\sim\mathcal{N}(0,\mho)$, $\sigma_n^2(x,\upsilon)=\Delta\psi(x,\upsilon)'\mho\Delta\psi(x,\upsilon)/\Delta q(x,\upsilon)^2$, and

$$\mathbb{Z}_n^{\pi}(x,\upsilon)=\frac{\Delta\psi(x,\upsilon)'}{\Delta q(x,\upsilon)\sigma_n(x,\upsilon)}\mathcal{Z}_n.$$

Lemma 4.1 in CC provides uniform Bahadur representation and uniform Gaussian process strong approximation

$$\sup_{(x,\upsilon)\in\Upsilon}\left|\frac{\sqrt{n}\left(\hat{\pi}(x,\upsilon)-\pi(x,\upsilon)\right)}{\hat{\sigma}(x,\upsilon)}-\mathbb{Z}_n^{\pi}(x,\upsilon)\right|=o_p(1).$$

Proof of Lemma 5: Note $\Delta S_i'=(0,\mathbf{0}'_{(d_x\times 1)},1,X_i')'$, $\beta=(a_0(\upsilon),a_1'(\upsilon),a_2(\upsilon),a_3'(\upsilon))'$, and $\hat{\beta}=(\hat{a}_0(\upsilon),\hat{a}_1'(\upsilon),\hat{a}_2(\upsilon),\hat{a}_3'(\upsilon))'$. We show that $(\upsilon,\beta)\mapsto\mathbb{G}_n\Delta m_i\chi_i=\sqrt{n}\sum_{i=1}^n\left(\Delta m_i\chi_i-\mathbb{E}\left[\Delta m_i\chi_i\right]\right)$ is stochastic equicontinuous over $\mathcal{V}\times\mathcal{B}$, with respect to the $L_2(P)$ pseudometric $\rho((\upsilon_1,\beta_1),(\upsilon_2,\beta_2))^2=\mathbb{E}\big[\big(\Delta m(X_i,\upsilon_1)(1(\Delta S_i'\beta_1\geq 0)-\Delta m(X_i,\upsilon_2)1(\Delta S_i'\beta_2\geq 0)\big)^2\big]$.

Following the proof of Theorem 3 in Section A.1.2 in the appendix of ACF, let $\mathcal{F}=\big\{1(\Delta S_i'\beta>0),\beta\in\mathcal{B}\big\}$ that is a VC subgraph class and hence a bounded Donsker class. $\mathcal{F}\Delta m(X,\upsilon)$ is Donsker with a square-integrable envelop $|\Delta m(X,\upsilon)|$ by Theorem 2.10.6 in Van der Vaart and

Wellner (1996).

By stochastic equicontinuity of $(v, \beta) \mapsto \mathbb{G}_n \Delta m_i \chi_i$, $n^{-1/2} \sum_{i=1}^n \Delta m_i \big(\hat{\chi}_i - \chi_i\big) = \sqrt{n}\mathbb{E}\big[\Delta m_i \big(\hat{\chi}_i - \chi_i\big)\big] + o_{p^*}(1) = \frac{\partial}{\partial \alpha}\mathbb{E}\big[\Delta m(X_i, v)1(\Delta S_i'\alpha \geq 0)\big]'\big|_{\alpha=\beta(v)} \times \sqrt{n}(\hat{\beta}(v) - \beta(v)) + o_{p^*}(1)$ uniformly over $v \in \mathcal{V}$, which follows from $\|\hat{\beta}(v) - \beta(v)\| = o_{p^*}(1)$, and resulting convergence with respect to the pseudometric $\sup_{v\in\mathcal{V}} \rho((v, \hat{\beta}(v)), (v, \beta(v)))^2 = o_p(1)$. The latter is from $\rho((v, \beta), (v, B))^2 = \mathbb{E}\big[\Delta m(X_i, v)^2(1(\Delta S_i'\beta \geq 0) - 1(\Delta S_i' B \geq 0))^2\big] = O\big(\frac{\partial}{\partial \alpha}\mathbb{E}\big[\Delta m(X_i, v)^2 1(\Delta S_i'\alpha \geq 0)\big]\big|'_{\alpha=\beta}(B - \beta)\big)$ for $\beta, B \in \mathcal{B}$, which we show below.

We can rewrite $1(\Delta S_i'\beta \geq 0) - 1(\Delta S_i' B \geq 0) = 1(\Delta S_i'\beta \geq 0, \Delta S_i' B < 0) - 1(\Delta S_i'\beta < 0, \Delta S_i' B \geq 0)$, and hence $\big(1(\Delta S_i'\beta \geq 0) - 1(\Delta S_i' B \geq 0)\big)^2 = 1(\Delta S_i'\beta \geq 0, \Delta S_i' B < 0) + 1(\Delta S_i'\beta < 0, \Delta S_i' B \geq 0)$. By symmetry, we focus on the second term. We can write $1(\Delta S_i'\beta < 0, \Delta S_i' B \geq 0) = (1(\Delta S_i' B \geq 0) - 1(\Delta S_i'\beta \geq 0))1(\Delta S_i'(B - \beta) \geq 0)$. Then $\mathbb{E}\big[\Delta m(X_i, v)^2(1(\Delta S_i' B \geq 0) - 1(\Delta S_i'\beta \geq 0))1(\Delta S_i'(B - \beta) \geq 0)\big] \leq \mathbb{E}\big[\Delta m(X_i, v)^2(1(\Delta S_i' B \geq 0) - 1(\Delta S_i'\beta \geq 0))\big] = \frac{\partial}{\partial \alpha}\mathbb{E}\big[\Delta m(X_i, v)^2 1(\Delta S_i'\alpha \geq 0)\big]\big|'_{\alpha=\bar{\beta}}(B - \beta)$, where $\bar{\beta}$ is between β and B by the mean value theorem.

$n^{-1/2} \sum_{i=1}^n l^{-1} \sum_{v\in V^{(l)}} \Delta m_i \big(\hat{\chi}_i - \chi_i\big) = l^{-1} \sum_{v\in V^{(l)}} \frac{\partial}{\partial \alpha}\mathbb{E}\big[\Delta m(X_i, v)1(\Delta S_i'\alpha \geq 0)\big]'\big|_{\alpha=\beta(v)} \times\sqrt{n}(\hat{\beta}(v) - \beta(v)) + o_{p^*}(1) = n^{-1/2} \sum_{j=1}^n \int_0^1 \frac{\partial}{\partial \alpha}\mathbb{E}\big[\Delta m(X_i, v)1(\Delta S_i'\alpha \geq 0)\big]'\big|_{\alpha=a(v)}\phi_j(v)dv + o_{p^*}(1)$ by Lemma 6.

The same arguments yield the result in 2. by replacing Δm with Δq.

Proof of Lemma 6: Let $\mathcal{V}(x) = \{v \in \mathcal{V} \;\; \Delta q(x, v) > 0\}$. The approximation error of Riemann sum is $\sup_{x\in\mathcal{X}} \big|l^{-1} \sum_{v\in V^{(l)}\cap\mathcal{V}(x)} f(x, v) - \int_{\mathcal{V}(x)} f(x, v)dv\big|$
$= O\big(\sup_{x\in\mathcal{X}} l^{-1} \sum_{v_j\in V^{(l)}} \big(\sup_{v\in(v_{j-1},v_j)} f(x, v) - \inf_{v\in(v_{j-1},v_j)} f(x, v)\big)\big)$
$= O\big(\sup_{x\in\mathcal{X}} l^{-1} \sup_{P\in\mathcal{P}} \sum_{j=0}^{n_P} \big| f(x, v_j) - f(x, v_{j-1})\big|\big) = O(l^{-1})$, where the set of all partitions $\mathcal{P} = \big\{P = \{v_0, \ldots, v_{n_P}\} \subset \mathcal{V}\big\}$.

Proof of Theorem 5: Decompose $\hat{\pi}^{DR,K} - \pi^{DR,K} = \sum_{k=1}^K \hat{\lambda}_k\hat{\pi}_k - \lambda_k\pi_k = \sum_{k=1}^K (\hat{\lambda}_k - \lambda_k)\pi_k + \lambda_k(\hat{\pi}_k - \pi_k) + O_p((\hat{\lambda}_k - \lambda_k)(\hat{\pi}_k - \pi_k))$.

Let $n_k = \sum_{i=1}^n D_i^k$. By the proof of Theorem 4, $\sum_{k=1}^K \lambda_k(\hat{\pi}_k - \pi_k) = \sum_{k=1}^K \lambda_k(n_k +$

$n_{k-1})^{-1}\sum_{i=1}^n(D_i^k + D_i^{k-1})R_i^k/B^k + o_p(n^{-1/2}) = n^{-1}\sum_{i=1}^n\sum_{k=1}^K \lambda_k \frac{(D_i^k+D_i^{k-1})R_i^k}{(p_k+p_{k-1})B^k} + o_p(n^{-1/2})$, where $R_i^k = R_{1i}^k + R_{2i}^k + R_{3i}$,

$$
\begin{aligned}
R_{1i}^k = \int_0^1 \Big(&\mathbb{E}(\partial_t m_1(X, q_1(X,v))S_1 - \partial_t m_0(X, q_0(X,v))S_0 - \pi^{DR}\Delta S)\\
&\mathrm{sgn}(X,v)(D^k + D^{k-1}) + \frac{\partial}{\partial\alpha}\mathbb{E}\big[\big(\Delta m(X,v) - \pi^{DR}\Delta q(X,v)\big)\big(1(\Delta S'\alpha \geq 0)\\
&- 1(\Delta S'\alpha \leq 0)\big)(D^k + D^{k-1})\big]\big|_{\alpha=a(v)}\Big)'\phi_i^k(v)dv/(p_k+p_{k-1}),\\
&\text{with } \phi_i^k(v) = \vartheta_k(v)^{-1}\big(1(T_i \leq S_i'a_k(v)) - v\big)S_i,\\
&\vartheta_k(v) = \mathbb{E}\Big[f_{T|X,Z}(S'a_k(v)|X,Z)SS'(D^k + D^{k-1})\Big]/(p_k+p_{k-1}),\\
&S_{1i} = (1, X_i', 1, X_i')', S_{0i} = (1, X_i', 0, \mathbf{0}'_{(d_x\times 1)})', \Delta S_i = S_{1i} - S_{0i},
\end{aligned}
$$

$$
\begin{aligned}
R_{2i}^k &= \mathcal{D}_k' G_k^{-1}\psi^J(X_i, T_i, Z_i)e_i,\\
&\quad\text{with } G_k = \mathbb{E}\Big[e^2\psi^J(X,T,Z)\psi^J(X,T,Z)'(D^k + D^{k-1})\Big]/(p_k+p_{k-1}),\\
&\quad\mathcal{D}_k = \int_0^1 \mathbb{E}\big[(\psi^J(X, q_1(X,v), 1) - \psi^J(X, q_0(X,v), 0))\mathrm{sgn}(X,v)\\
&\qquad\times (D^k + D^{k-1})\big]dv/(p_k+p_{k-1}),\\
R_{3i} &= \int_0^1 \big(\Delta m(X_i,v) - \pi^{DR}\Delta q(X_i,v)\big)\,\mathrm{sgn}(X_i,v)dv,\\
B^k &= \int_0^1 \mathbb{E}\Big[|\Delta q(X,v)|1(|\Delta q(X,v)| \geq 0)(D^k + D^{k-1})\Big]dv/(p_k+p_{k-1}). \qquad \text{(S.13)}
\end{aligned}
$$

Next we analyze $\sum_{k=1}^K(\hat\lambda_k - \lambda_k)\pi_k$. Let $\mathsf{A}_k = Q_k P_k$, where $Q_k = q_k - q_{k-1}$ and $P_k = \sum_{l=k}^K p_l(q_l - \mathbb{E}[T])$. Let $\lambda_k = \mathsf{A}_k/\mathsf{B}$, where $\mathsf{B} = \sum_{k=1}^K \mathsf{A}_k$. So $\pi^{DR,K} = \sum_{k=1}^K \pi_k\mathsf{A}_k/\mathsf{B}$. Then $\sum_{k=1}^K(\hat\lambda_k - \lambda_k)\pi_k = \sum_{k=1}^K\big\{(\hat{\mathsf{A}}_k - \mathsf{A}_k)/\mathsf{B} - (\hat{\mathsf{B}} - \mathsf{B})\mathsf{A}_k/\mathsf{B}^2\big\}\pi_k + o_p(n^{-1/2}) = \sum_{k=1}^K(\hat{\mathsf{A}}_k - \mathsf{A}_k)(\pi_k - \pi^{DR,K})/\mathsf{B} + o_p(n^{-1/2})$.

Decompose $\hat{\mathsf{A}}_k - \mathsf{A}_k = (\hat Q_k - Q_k)P_k + (\hat P_k - P_k)Q_k + o_p(n^{-1/2})$. It is straightforward to show that $\hat q_k - q_k = n^{-1}\sum_{i=1}^n\big\{(T_iD_i^k - \mathbb{E}\big[T_iD_i^k\big])/p_k - (D_i^k - p_k)q_k/p_k\big\} + o_p(n^{-1/2}) = n^{-1}\sum_{i=1}^n(T_i - q_k)D_i^k/p_k + o_p(n^{-1/2})$.

$\hat{P}_k - P_k = \sum_{l=k}^{K}\left\{(\hat{p}_l - p_l)(q_l - \mathbb{E}[T]) + p_l\big(\hat{q}_l - q_l - \bar{T} + \mathbb{E}[T]\big)\right\} + o_p(n^{-1/2}) = n^{-1}\sum_{i=1}^{n}\sum_{l=k}^{K}\left\{(D_{li} - p_l)(q_l - \mathbb{E}[T]) + p_l\big((T_i - q_l)D_{li}/p_l - T_i + \mathbb{E}[T]\big)\right\} + o_p(n^{-1/2}) = n^{-1}\sum_{i=1}^{n}\sum_{l=k}^{K}\big(D_{li} - p_l\big)\big(T_i - \mathbb{E}[T]\big) - P_k + o_p(n^{-1/2})$.

Therefore $\sum_{k=1}^{K}(\hat{\lambda}_k - \lambda_k)\pi_k = \sum_{k=1}^{K}(\hat{\mathsf{A}}_k - \mathsf{A}_k)(\pi_k - \pi^{DR,K})/\mathsf{B} + o_p(n^{-1/2}) = n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K} R_{4ki} + o_p(n^{-1/2})$, where

$$R_{4i}^k = \left\{\left((T_i - q_k)\frac{D_i^k}{p_k} - (T_i - q_{k-1})\frac{D_i^{k-1}}{p_{k-1}}\right)\sum_{l=k}^{K} p_l(q_l - \mathbb{E}[T]) + (q_k - q_{k-1}) (T_i - \mathbb{E}[T])\sum_{l=k}^{K}\big(D_{li} - p_l\big)\right\}\frac{\pi_k - \pi^{DR,K}}{\sum_{k=1}^{K}(q_k - q_{k-1})\sum_{l=k}^{K} p_l(q_l - \mathbb{E}[T]).}. \tag{S.14}$$

By R_i^k given in (S.13) and R_{4i}^k given in (S.14), we obtain the influence function

$$R_{Ki} = \sum_{k=1}^{K}\lambda_k\frac{(D_i^k + D_i^{k-1})R_i^k}{(p_k + p_{k-1})B^k} + R_{4i}^k. \tag{S.15}$$

Asymptotic normality follows the same arguments in the proof of Theorem 4 with the following modifications. The law of iterated expectations yields $\sigma_{Kn}^2 = \sigma_{K1}^2 + \sigma_{K2n}^2 + \sigma_{K3}^2$, where $\sigma_{K1}^2 = \mathbb{E}\left[\left(\sum_{k=1}^{K}\lambda_k\frac{(D_i^k+D_i^{k-1})R_{1i}^k}{(p_k+p_{k-1})B^k} + R_{4i}^k\right)^2\right]$, $\sigma_{K2n}^2 = \mathbb{E}\left[\left(\sum_{k=1}^{K}\lambda_k\frac{(D_i^k+D_i^{k-1})R_{2i}^k}{(p_k+p_{k-1})B^k}\right)^2\right]$, and $\sigma_{K3}^2 = \mathbb{E}\left[\left(\sum_{k=1}^{K}\lambda_k\frac{(D_i^k+D_i^{k-1})R_{3i}}{(p_k+p_{k-1})B^k}\right)^2\right]$.

S.6 Estimation and inference without covariates

This section presents nonparametric estimation and inference for the case without covariates, building on the results of the general case with covariates discussed in Section 4 in the main text.

Consider the quantile regression model for the conditional u quantile of T given $Z = z$, defined as $q_z(u) = a_0(u) + za_1(u)$. Additionally, let the nonparametric model for the conditional mean of Y given Z and T be $m_z(t) = g_0(t) + zg_1(t)$, where g_z, $z = 0, 1$, are some unknown functions.

Step 1. Estimate the first-stage conditional treatment quantiles $q_z(u)$: $\widehat{q}_z(u) = \widehat{a}_0(u) + z\hat{a}_1(u)$ for $u \in \mathsf{U}^{(l)}$, where $\mathsf{U}^{(l)} = \{u_1, u_2, ..., u_l\}$ is the set of equally spaced quantiles over $(0, 1)$.

Then $\Delta\widehat{q}(u) = \hat{a}_1(u)$.

Step 2. Estimate the conditional mean function $m_z(t)$ by a series estimator: $\widehat{m}_z(t) = \widehat{g}_0(t) + z\widehat{g}_1(t)$. Let $\Delta\hat{m}(u) = \widehat{m}_1(\widehat{q}_1(u)) - \widehat{m}_0(\widehat{q}_0(u))$.

Step 3. For $u \in \mathsf{U}^{(l)}$, the plug-in estimator of $\tau(u)$ is $\hat{\tau}(u) = \Delta\hat{m}(u)/\Delta\widehat{q}(u)$.

Estimate τ^{DR}: $\widehat{\tau}^{DR} = \sum_{u\in\mathcal{U}^{(l)}} \widehat{\tau}(u)\widehat{\omega}(u)$, where $\widehat{\omega}(u) = |\Delta\widehat{q}(u)|/\sum_{u\in\mathcal{U}^{(l)}} |\Delta\widehat{q}(u)|$.

For the series estimator, let $\psi^J(t,z) = \big(\psi_{J1}(t), ..., \psi_{JJ}(t), z\psi_{J1}(t), ..., z\psi_{JJ}(t)\big)'$, a $2J \times 1$ vector, and $\Psi = (\psi^J(T_1, Z_1), ..., \psi^J(T_n, Z_n))'$, a $n \times 2J$ matrix. Then the series coefficient estimate is $\hat{c} = (\Psi'\Psi)^{-1}\Psi'(Y_1, ..., Y_n)'$, and a series least squares estimator of $m_z(t)$ is $\hat{m}_z(t) = \psi^J(t,z)'\hat{c}$.

Let the sieve variance estimator for $\hat{\tau}(u)$ be $\hat{\sigma}^2(u) = \Delta\hat{\psi}(u)'\hat{\mho}\Delta\hat{\psi}(u)/\Delta\hat{q}(u)^2$, where $\Delta\hat{\psi}(u) = \psi^J(\hat{q}_1(u), 1) - \psi^J(\hat{q}_0(u), 0)$ and $\hat{\mho}$ given in Section S.7 is a consistent estimator for $\mho$ defined in S.3 by removing X_i.

Theorem 7. *Let Assumptions A1-A3 hold without X. Then $\sqrt{n}(\hat{\tau}(u) - \pi(u))/\hat{\sigma}(u) \xrightarrow{d} \mathcal{N}(0,1)$ uniformly for $u \in \Upsilon = \{u \in \mathcal{U} \ \ |\Delta q(u)| \geq 0\}$.*

Similarly as Theorem 4, Theorem 8 shows that the influence function of $\hat{\tau}^{DR}$ is given by $R_i/B = (R_{1i} + R_{2i})/B$. The exact formulas of R_{ki}, $k = 1, 2$, are given in (S.17) in the proof.

Theorem 8. *Let Assumptions A1, A2, and A3 hold without X. Let $\sqrt{n}l^{-1} = o(1)$. Then $\sqrt{n}\big(\hat{\tau}^{DR} - \tau^{DR}\big)/\hat{\sigma} = n^{-1/2}\sum_{i=1}^{n} R_i/(B\sigma_n) + o_p(1) \xrightarrow{d} \mathcal{N}(0,1)$.*

Notation for Theorems 7 and 8:

$$\phi_i(u) = \vartheta(u)^{-1}\big(1(T_i \leq S_i'a(u)) - v\big)S_i$$

$$S_i = (1, Z)',\, S_1 = (1,1)',\, S_0 = (1,0)',\, \Delta S = S_1 - S_0 = (0,1)'$$

$$\partial_t m_z(q_z(u)) = \frac{\partial}{\partial t} m_z(t)|_{t=q_z(u)}$$

$$q_z = q_z(u),\, \hat{q}_z = \hat{q}_z(u)$$

$$\Delta q = \Delta q(u) = q_1 - q_0 = (S_1 - S_0)'a(u) = \Delta S'a(u)$$
$$\Delta \hat{q} = \Delta \hat{q}(u) = \hat{q}_1 - \hat{q}_0 = (S_1 - S_0)'\hat{a}(u) = \Delta S'\hat{a}(u)$$
$$\Delta \psi = \Delta \psi(u) = \psi^J(q_1(u), 1) - \psi^J(q_0(u), 0)$$
$$\Delta \hat{\psi} = \Delta \hat{\psi}(u) = \psi^J(\hat{q}_1(u), 1) - \psi^J(\hat{q}_0(u), 0)$$
$$\Delta m = \Delta m(u) = m_1(q_1(u)) - m_0(q_0(u))$$
$$\Delta \hat{m} = \Delta \hat{m}(u) = \hat{m}_1(\hat{q}_1(u)) - \hat{m}_0(\hat{q}_0(u)) = \Delta \hat{\psi}'\hat{c}$$
$$\Delta \check{m} = \Delta \check{m}(u) = \hat{m}_1(q_1(u)) - \hat{m}_0(q_0(u)) = \Delta \psi'\hat{c}$$
$$\chi = \chi(u) = 1(|\Delta q(u)| \geq 0|)$$
$$\chi^{\pm} = \chi^{\pm}(u) = 1(\pm\Delta q(u) \geq 0|)$$

Lemma 7 shows that the estimation error of the sign function is $o_p(1/\sqrt{n})$. In contrast, Lemma 5 shows that with X, the estimation error of the sign function depends on the tail distribution of $\Delta m(X, T)$.

Lemma 7. *Let Assumption A1 hold. Let* $\sqrt{n}l^{-1} = o(1)$. *Then* $l^{-1}\sum_{u \in \mathsf{U}^{(l)}} \Delta m(u)\big(\hat{\chi}^+(u) - \chi^+(u)\big) = o_p(1/\sqrt{n})$ *and* $l^{-1}\sum_{u \in \mathsf{U}^{(l)}} \Delta q(u)\big(\hat{\chi}^+(u) - \chi^+(u)\big) = o_p(1/\sqrt{n})$.

Proof of Theorem 7: The proof follows the proofs of Theorem 6 by removing X. So the estimation error of $\hat{q}_z(u)$ in Step 1 is of smaller order than the estimation error in Step 2, i.e., the first-order asymptotic distribution of $\hat{\tau}(u)$ is as if $q_z(u)$ was known. Particularly for Step 2, define $\mathcal{Z}_n \sim \mathcal{N}(0, \mho)$ and $\mathbb{Z}_n^{\tau}(u) = \frac{\Delta \psi(u)'}{\Delta q(u)\sigma_n(u)}\mathcal{Z}_n$. Lemma 4.1 in CC provides uniform Bahadur representation and uniform Gaussian process strong approximation

$$\sup_{u \in \Upsilon}\left|\frac{\sqrt{n}\left(\hat{\tau}(u) - \tau(u)\right)}{\hat{\sigma}(u)} - \mathbb{Z}_n^{\tau}(u)\right| = o_p(1).$$

Proof of Theorem 8: The proof follows the proof of Theorem 4 for τ^{DR} by removing X. In particular, define A_+ and A_- as $A_{\pm} = \int_0^1 \Delta m(u)\chi^{\pm}(u)du$. So $A = A_+ - A_- =$

$\int_0^1 \Delta m(u)/\Delta q(u)\big(\Delta q(u)1(\Delta q(u) \geq 0) - \Delta q(u)1(\Delta q(u) \leq 0)\big)du = \int_0^1 \tau(u)|\Delta q(u)|1(|\Delta q(u)| \geq 0)du$.

Define B_+ and B_- as $B_\pm = \int_0^1 \Delta q(u)\chi^\pm(u)du$. By a similar argument as A, we can show that $B = B_+ - B_-$. Therefore, $\tau^{DR} = A/B$ and $\tau_\pm^{DR} = A_\pm/B_\pm$. Linearize $\hat{\tau}^{DR} - \tau^{DR} = (\hat{A} - A)/B - (\hat{B} - B)\tau/B + O_p\left(|\hat{A} - A||\hat{B} - B|/B^2 + |\hat{B} - B|^2/B^2\right)$.

The proof focuses on $\hat{A}_+$, the estimator of A_+. The same arguments apply to $\hat{B}_+$, the estimator of B_+. The same arguments apply to $\hat{\tau}_-^{DR}$ and hence $\hat{\tau}^{DR}$. Write $\hat{\tau}_+^{DR} = \hat{A}_+/\hat{B}_+$, where $\hat{A}_+ = \frac{1}{l}\sum_{u \in \mathsf{U}^{(l)}} \Delta\hat{m}(u)\hat{\chi}^+(u)$ and $\hat{B}_+ = \frac{1}{l}\sum_{u \in \mathsf{U}^{(l)}} \Delta\hat{q}(u)\hat{\chi}^+(u)$.

In the following, we suppress the subscripts of $+$ and superscripts of DR for expositional simplicity. Linearize $\hat{\tau} - \tau = (\hat{A} - A)/B - (\hat{B} - B)\tau/B + O_p\left(|\hat{A} - A||\hat{B} - B|/B^2 + |\hat{B} - B|^2/B^2\right)$.

Let $\tilde{A} = l^{-1}\sum_{u \in \mathsf{U}^{(l)}} \Delta\hat{m}(u)\chi(u)$ for a known sign function. Decompose $\hat{A} - A = \hat{A} - \tilde{A} + \tilde{A} - A$. The estimation error in $\Delta\hat{m}$.

$$\tilde{A} - A = \frac{1}{l}\sum_{u \in \mathsf{U}^{(l)}} \big(\Delta\hat{m}(u) - \Delta m(u)\big)\chi(u) + \frac{1}{l}\sum_{u \in \mathsf{U}^{(l)}} \Delta m(u)\chi(u) - A, \tag{S.16}$$

where the second term is $o_p(n^{-1/2})$ by Lemma 6 and assuming $\sqrt{n}l^{-1} = o(1)$. We focus on the first term next. Decompose $\Delta\hat{m} - \Delta m = \big(\Delta\hat{m} - \Delta\check{m}\big) + \big(\Delta\check{m} - \Delta m_i\big)$. The first part is for Step 1 estimation error, and the second part is for Step 2 estimation error.

Step 1 Theorem 3 in ACF shows that $\hat{a}(u) - a(u) = n^{-1}\sum_{i=1}^n \phi_i(u) + o_p(n^{-1/2})$ uniformly over $u \in \mathsf{U}^{(l)}$ and converges in distribution to a zero mean Gaussian process indexed by u. Decompose

$$\begin{aligned}
&\Delta\hat{m} - \Delta\check{m} \\
&= m_1(\hat{q}_1) - m_1(q_1) - (m_0(\hat{q}_0) - m_0(q_0)) + so1 \\
&= \partial_t m_1(q_1)(\hat{q}_1 - q_1) - \partial_t m_0(q_0)(\hat{q}_0 - q_0) + so1 + so2 \\
&= \partial_t m_1(q_1)S_1(\hat{a}(u) - a(u)) - \partial_t m_0(q_0)S_0(\hat{a}(u) - a(u)) + so1 + so2,
\end{aligned}$$

where (We suppress the subscript i for simplicity.)

$$
\begin{aligned}
so1 &= \hat{m}_1(\hat{q}_1) - m_1(\hat{q}_1) - \big(\hat{m}_0(\hat{q}_0) - m_0(\hat{q}_0)\big) - \big(\hat{m}_1(q_1) - m_1(q_1)\big) \\
&\quad + \big(\hat{m}_0(q_0) - m_0(q_0)\big) \\
&= O_p\left(\|\partial_t \hat{m}_z - \partial_t m_z\|_\infty \|\hat{q}_z - q_z\|_\infty\right), \\
so2 &= O_p\left(\partial_t^2 m_1\big(\hat{q}_1 - q_1\big)^2 + \partial_t^2 m_0\big(\hat{q}_0 - q_0\big)^2\right) = O_p(\|\hat{q}_z - q_z\|_\infty^2),
\end{aligned}
$$

as $\partial_t^2 m_z$ is uniformly bounded by Assumption A3. ACF and Corollary 3.1(ii) in CC implies that $so1 + so2 = O_p(\|\hat{q}_z - q_z\|_\infty \|\partial_t \hat{m}_z - \partial_t m_z\|_\infty + \|\hat{q}_z - q_z\|_\infty^2) = O_p(n^{-1/2}(J^{-(p-1)} + J\sqrt{(J\log J)/n}) + n^{-1}) = o_p(n^{-1/2})$ uniformly over $u \in \mathcal{U}$, by assuming $J\sqrt{(J\log J)/n} = o(1)$ and $p > 1$. Then $\sqrt{n}l^{-1}\sum_{u\in \mathsf{U}^{(l)}}(\Delta\hat{m} - \Delta\check{m})\chi$

$$
\begin{aligned}
&= \frac{1}{l}\sum_{u\in \mathsf{U}^{(l)}} (\partial_t m_1(q_1) S_1' \chi \sqrt{n}(\hat{a}(u) - a(u)) - \partial_t m_0(q_0) S_0' \chi \sqrt{n}(\hat{a}(u) - a(u)) + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{j=1}^{n} \frac{1}{l}\sum_{u\in \mathsf{U}^{(l)}} \left(\partial_t m_1(q_1) S_1 - \partial_t m_0(q_0) S_0\right)' \chi \phi_j(u) + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{j=1}^{n} \int_0^1 \left(\partial_t m_1(q_1) S_1 - \partial_t m_0(q_0) S_0\right)' \chi \phi_j(u) du + o_p(1),
\end{aligned}
$$

where the second equality is by ACF, and the last equality is by Lemma 6 and $\sqrt{n}l^{-1} = o(1)$.

By Lemma 6 and assuming $\sqrt{n}l^{-1} = o(1)$, $l^{-1}\sum_{u\in \mathsf{U}^{(l)}}(\Delta\check{m} - \Delta m)\chi = \int_0^1 \Delta\check{m}\chi du - A + o_p(n^{-1/2})$.

Note that A is based on a linear functional of m, $L(m) = \int_0^1 m_z(q_z(u))1(\Delta q(u) > 0)du$. So we use the results on linear functionals of a sieve estimator in CC. Let $\sigma_{A2n}^2 = \mathbb{E}\left[R_{A2i}^2\right]$, where $R_{A2i} = \mathcal{D}^{+\prime} G^{-1}\psi^J(T_i, Z_i)e_i$ and $\mathcal{D}^+ = \int_0^1 \Delta\psi^J(u)\chi^+(u)du$, with a consistent estimator $\hat{\sigma}_{A2}^2$. Lemma 4.1 in CC provides

$$
\left|\frac{\sqrt{n}}{\hat{\sigma}_{A2}}\left(\int_0^1 \Delta\check{m}\chi du - A\right) - \frac{1}{\sigma_{A2n}\sqrt{n}}\sum_{i=1}^{n} R_{A2i}\right| = o_p(1).
$$

The estimation error from the sign function $\hat{A} - \tilde{A} = n^{-1}\sum_{i=1}^{n} l^{-1}\sum_{u\in\mathsf{U}^{(l)}} \Delta m(u)\big(\hat{\chi}(u) - \chi(u)\big) + o_p(1/\sqrt{n})$ by $n^{-1}\sum_{i=1}^{n} l^{-1}\sum_{u\in\mathsf{U}^{(l)}} \big(\Delta\hat{m}(u) - \Delta m(u)\big)\big(\hat{\chi}(u) - \chi(u)\big) = O_p\Big(\|\Delta\hat{m} - \Delta m\|_\infty\|\hat{q}_z - q_z\|_\infty\Big) = o_p(n^{-1/2})$. Together with Lemma 7(i), $\big|\sqrt{n}(\hat{A} - A) - n^{-1/2}\sum_{i=1}^{n} R_{Ai}\big| = o_p(1)$, where $R_{Ai} = R_{A1i} + R_{A2i}$ with

$$R_{A1i} = \int_0^1 (\partial_t m_1(q_1)S_1 - \partial_t m_0(q_0)S_0)'\,\chi^+(u)\phi_i(u)du,$$

By the similar arguments as for A in (S.16),

$$\tilde{B} - B = \frac{1}{l}\sum_{u\in\mathsf{U}^{(l)}} \big(\Delta\hat{q}(u) - \Delta q(u)\big)\,\chi(u) + \frac{1}{l}\sum_{u\in\mathsf{U}^{(l)}} \Delta q(u)\chi(u) - B,$$

where the second term is $\int_0^1 \Delta q(u)\chi(u)du - B + o_p(n^{-1/2})$ by Lemma 6, and the first term is $l^{-1}\sum_{u\in\mathsf{U}^{(l)}} \Delta S'\big(\hat{a}(u) - a(u)\big)\chi = n^{-1}\sum_{j=1}^{n} l^{-1}\sum_{u\in\mathsf{U}^{(l)}} \chi\,\Delta S'\phi_j(u) + o_p(n^{-1/2})$
$= n^{-1}\sum_{j=1}^{n}\int_0^1 \Delta S'\chi\phi_j(u)du + o_p(n^{-1/2})$, by ACF and Lemma 6.

Together with Lemma 7(ii), we obtain $\big|\sqrt{n}(\hat{B} - B) - n^{-1/2}\sum_{i=1}^{n} R_{Bi}\big| = o_p(1)$, where

$$R_{Bi} = \int_0^1 \left(\Delta S'\chi^+(u) + \frac{\partial}{\partial\alpha}\left[\Delta q(u)1(\Delta S'\alpha \geq 0)\right]'\big|_{\alpha=a(u)}\right)\phi_i(u)du.$$

By a linearization for $\hat{\tau}_+^{DR}$, $\hat{\tau}_+^{DR} - \tau_+^{DR} = \frac{\hat{A}_+}{\hat{B}_+} - \frac{A_+}{B_+} = \frac{\hat{A}_+ - A_+}{B_+} - \frac{\tau_+^{DR}}{B_+}(\hat{B}_+ - B_+) + o_p(n^{-1/2})$. Therefore, we define $R_i^+ = R_{Ai} - \tau_+^{DR}R_{Bi} = R_{1i}^+ + R_{2i}^+ + R_{3i}^+$, where $R_{1i}^+ = R_{A1i} - \tau_+^{DR}R_{B1i}$ and $R_{2i}^+ = R_{A2i}$. That is,

$$R_{1i}^+ = \int_0^1 \big(\partial_t m_1(q_1(u))S_1 - \partial_t m_0(q_0(u))S_0 - \tau_+^{DR}\Delta S\big)'\,\chi^+(u)\phi_i(u)du,$$
$$R_{2i}^+ = \mathcal{D}^{+'}G^{-1}\psi^J(T_i, Z_i)e_i, \text{ with } \mathcal{D}^+ = \int_0^1 \Delta\psi^J(u)\chi^+(u)du.$$

Then we obtain $\hat{\tau}_+^{DR} - \tau_+^{DR} = n^{-1}\sum_{i=1}^{n}\big(R_{Ai} - \tau_+^{DR}R_{Bi}\big)/B_+ + o_p(n^{-1/2}) = n^{-1}\sum_{i=1}^{n} R_i^+/B_+ + o_p(n^{-1/2})$.

Asymptotic normality We suppress the subscripts of $+$ and superscripts of DR for expositional simplicity. Because R_{2i} depends on (Y_i, T_i) and R_{1i} depends on T_i, the law of iterated expectations yields $\sigma_n^2 = \left(\mathbb{E}\left[R_{1i}^2\right] + \mathbb{E}\left[R_{2i}^2\right]\right)/B^2 = (\sigma_1^2 + \sigma_{2n}^2)/B^2$.

We will show the Bahadur representation that

$$
\begin{aligned}
&\left|\frac{\sqrt{n}(\hat{\tau} - \tau)}{\hat{\sigma}} - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{R_i}{B\sigma_n}\right| \\
&\leq \left|\frac{\sqrt{n}(\hat{\tau} - \tau)}{\sigma_n} - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{R_i}{B\sigma_n}\right| + \left|\frac{\sqrt{n}(\hat{\tau} - \tau)}{\sigma_n}\left(\frac{\sigma_n}{\hat{\sigma}} - 1\right)\right| = o_p(1)
\end{aligned}
$$

by (i) $n^{-1/2}\sum_{i=1}^{n} R_i/(B\sigma_n) \xrightarrow{d} \mathcal{N}(0, 1)$, and (ii) $|\sigma_n/\hat{\sigma} - 1| = o_p(1)$, as shown below.

(i) Asymptotic normality will follow from the Lyapunov central limit theorem with the third absolute moment, $n^{-1/2}\mathbb{E}|R_i|^3/(B\sigma_n)^3 \to 0$, since $\{R_i\}_{i=1}^n$ are independent across i, with mean zero and variance 1. By the assumed conditions, it is straightforward to show that $n^{-1/2}\mathbb{E}|R_{1i}|^3/(B\sigma_1)^3 \to 0$. We show below that $n^{-1/2}\mathbb{E}|R_{2i}|^3/(B\sigma_{2n})^3 \to 0$. Then it implies that all the cross-product terms $n^{-1/2}\mathbb{E}|R_{1i}R_{2i}|/(B\sigma_n)^3 \to 0$ and $n^{-1/2}\mathbb{E}|R_{ji}^2 R_{ki}|/(B\sigma_n)^3 \to 0$ for $j, k = 1, 2,\ j \neq k$.

Denote as $\psi = \psi^J(T_i, Z_i)$. By Assumption A2.2(ii),

$$
\sigma_{2n}^2 = \mathbb{E}\left[R_{2i}^2\right]/B^2 = \mathbb{E}\left[(\mathcal{D}'G^{-1}\psi)^2 e_i^2\right]/B^2 \geq \mathbb{E}\left[(\mathcal{D}'G^{-1}\psi)^2\right]\underline{\sigma}^2/B^2 = \mathcal{D}'G^{-1}\mathcal{D}\underline{\sigma}^2/B^2.
$$

By the Schwarz inequality and Assumption A2.3(ii),

$$
\frac{(\mathcal{D}'G^{-1}\psi)^2}{\sigma_{2n}^2} \leq \frac{(\mathcal{D}'G^{-1}\mathcal{D}')(\psi'G^{-1}\psi)}{\sigma_{2n}^2} \leq \frac{\zeta^2}{\underline{\sigma}^2}.
$$

Then by Assumption A2.2(iii),

$$\begin{aligned}
\frac{1}{\sqrt{n}}\mathbb{E}\left[\frac{|R_{2i}|^3}{B^3\sigma_{2n}^3}\right] &= \frac{1}{\sqrt{n}}\mathbb{E}\left[\frac{|\mathcal{D}'G^{-1}\psi e_i|^3}{B^3\sigma_{2n}^3}\right] \\
&= \frac{1}{\sqrt{n}}\mathbb{E}\left[\frac{(\mathcal{D}'G^{-1}\psi)^2}{B^3\sigma_{2n}^2}\frac{|\mathcal{D}'G^{-1}\psi|}{\sigma_{2n}}\mathbb{E}\left[|e_i|^3|T_i, Z_i\right]\right] \\
&\leq \frac{\zeta}{\sqrt{n}B^3\underline{\sigma}^3}\sup_{t,z}\mathbb{E}\left[|e_i|^3|X_i = T_i = t, Z_i = z\right] = O\left(\frac{\zeta}{\sqrt{n}}\right) = o(1).
\end{aligned}$$

(ii) It is straightforward that $\hat{\sigma}_1^2 = n^{-1}\sum_{i=1}^n \hat{R}_{1i}^2/\hat{B}^2 \xrightarrow{p} \sigma_1^2 = \mathbb{E}\left[R_{1i}^2\right]/B^2$ and $\hat{\sigma}_3^2 \xrightarrow{p} \sigma_3^2$. The same arguments in Lemma G.4 in CC give $|\sigma_{2n}/\hat{\sigma}_1 - 1| = O_p(\delta_{V,n}) = o_p(1)$. So $|\sigma_n/\hat{\sigma} - 1| = o_p(1)$.

By (i) that $n^{-1/2}\sum_{i=1}^n R_i/(B\sigma_n) = O_p(1)$ and (ii), the second term $\left|\frac{\sqrt{n}(\hat{\tau}-\tau)}{\hat{\sigma}}\left(\frac{\hat{\sigma}}{\sigma_n} - 1\right)\right| = O_p(1)o_p(1) = o_p(1)$. We then obtain the Bahadur representation. The asymptotic normality follows from the result (i).

Therefore, we obtain that when $B_+ > 0$, $\sqrt{n}\left(\hat{\tau}_+^{DR} - \tau_+^{DR}\right)/\hat{\sigma}_{n+} = n^{-1/2}\sum_{i=1}^n R_i^+ /(B_+\sigma_{n+}) + o_p(1) \xrightarrow{d} \mathcal{N}(0,1)$, where $\hat{\sigma}_+^2$ is a consistent estimator of $\sigma_{n+}^2 = \mathbb{E}\left[R_i^{+2}\right]/B_+^2$.

For τ_-^{DR}, define

$$\begin{aligned}
R_{1i}^- = \int_0^1 \Bigg(&\mathbb{E}\left[\left(\partial_t m_1(q_1(u))S_1 - \partial_t m_0(q_0(u))S_0 - \tau_-^{DR}\Delta S\right)\chi^-(u)\right] \\
&+ \frac{\partial}{\partial\alpha}\mathbb{E}\left[\left(\Delta m(u) - \tau_+^{DR}\Delta q(u)\right)1(\Delta S'\alpha \leq 0)\right]\Big|_{\alpha=a(u)}\Bigg)'\phi_i(u)du.
\end{aligned}$$

Define R_i^- as R_i^+ by replacing $+$ with $-$ in all the components in R_i^+. By the same arguments for τ_+^{DR}, we obtain that when $B_- > 0$, $\sqrt{n}\left(\hat{\tau}_-^{DR} - \tau_-^{DR}\right)/\hat{\sigma}_- = n^{-1/2}\sum_{i=1}^n R_i^-/(B_-\sigma_{n-}) + o_p(1) \xrightarrow{d} \mathcal{N}(0,1)$, where $\hat{\sigma}_-^2$ is a consistent estimator of $\sigma_{n-}^2 = \mathbb{E}\left[R_i^{-2}\right]/B_-^2$, such that $|\sigma_{n-}/\hat{\sigma}_- - 1| = o_p(1)$.

For τ^{DR}, the same linearization yields $\hat{\tau}^{DR} - \tau^{DR} = (\hat{A} - A)/B - (\hat{B} - B)\tau^{DR}/B + O_p\big(|\hat{A} - A||\hat{B} - B|/B^2 + |\hat{B} - B|^2/B^2\big)$. Let $R_i = R_i^+ - R_i^- = R_{1i} + R_{2i} + R_{3i}$, where $R_{li} = R_{li}^+ - R_{li}^-$

for $l = 1, 2$ by replacing τ_+^{DR} and τ_-^{DR} with τ^{DR}. Specifically, let $\mathrm{sgn}(u) = 1(\Delta q(u) \geq 0) - 1(\Delta q(u) \leq 0)$,

$$\begin{aligned}
R_{1i} &= \int_0^1 \mathbb{E}\big[(\partial_t m_1(q_1(u))S_1 - \partial_t m_0(q_0(u))S_0 - \tau^{DR}\Delta S)\mathrm{sgn}(u)\big]'\phi_i(u)du, \text{with} \\
&\quad \phi_i(u) = \vartheta(u)^{-1}\big(1(T_i \leq S_i'a(u)) - v\big)S_i, S_1 = (1,1)', S_0 = (1,0)', \Delta S = S_1 - S_0 = (0,1)', \\
R_{2i} &= \mathcal{D}'G^{-1}\psi^J(T_i, Z_i)e_i, \text{with } \mathcal{D} = \int_0^1 \mathbb{E}\left[(\psi^J(q_1(u),1) - \psi^J(q_0(u),0))\mathrm{sgn}(u)\right]du. \quad \text{(S.17)}
\end{aligned}$$

Proof of Lemma 7: By Markov inequality, for any $\delta > 0$, $P(|\sqrt{n}\int_0^1 \Delta m(u)(1(\Delta S'\hat{\alpha} \geq 0) - 1(\Delta S'\alpha \geq 0))| \geq \delta) \leq \mathbb{E}n\int_0^1 |\Delta m(u)|^2(1(\Delta S'\hat{\alpha} \geq 0) - 1(\Delta S'^2/\delta^2 \leq \mathbb{E}n\int_0^1 |\Delta m(u)|^2$ $\times(1(\Delta S'\hat{\alpha} \geq 0, \Delta S'\alpha < 0) - 1(\Delta S'\hat{\alpha} < 0, \Delta S'^2/\delta^2 \leq \mathbb{E}n\int_0^1 |\Delta m(u)|^2(1(\Delta S'\hat{\alpha} \geq 0, \Delta S'\alpha < 0) + 1(\Delta S'\hat{\alpha} < 0, \Delta S'\alpha \geq 0))/\delta^2 \leq$ $\mathbb{E}n\int_0^1 |\Delta m(u)|^2 1(|\Delta S'(\hat{\alpha}-\alpha)| > |\Delta S'\alpha|)/\delta^2 \leq n\int_0^1 |\Delta m(u)|^2\mathbb{E}1(|\Delta S'\sqrt{n}(\hat{\alpha}-\alpha)| > \sqrt{n}|\Delta S'\alpha|)/\delta^2$ $= O_p(n2\Phi(-\sqrt{n}|\Delta S'\alpha|)) = o_p(1)$, where Φ is the CDF of $\mathcal{N}(0,1)$, assuming $\int_0^1 |\Delta m(u)|^2 du < \infty$.

The same arguments yield the result for Δq by replacing Δm with Δq.

S.7 Variance Estimation

For convenience, we first collect the relevant notations and then discuss implementation details.

S.7.1 Notation:

Let $\phi_i(v) = \vartheta(v)^{-1}\big(1(T_i \leq S_i'a(v)) - v\big)S_i$. Let the positive sign function $\chi^+(x, v) = 1(\Delta q(x, v) \geq 0)$. Let $S_{1i} = (1, X_i', 1, X_i')'$, $S_{0i} = (1, X_i', 0, \mathbf{0}'_{(d_x \times 1)})'$, $\Delta S_i = S_{1i} - S_{0i}$, $\partial_t m_z(X, q_z) = \frac{\partial}{\partial t}m_z(X,t)|_{t=q_z(X,v)}$.

$$
\begin{aligned}
R_{1i}^{+} = & \int_0^1 \Bigg(\mathbb{E}\left[\left(\partial_t m_1(X, q_1(X, \upsilon)) S_1 - \partial_t m_0(X, q_0(X, \upsilon)) S_0 - \pi_+^{DR} \Delta S \right) \chi^+(X, \upsilon) \right] \\
& + \frac{\partial}{\partial \alpha} \mathbb{E}\left[\left(\Delta m(X, \upsilon) - \pi_+^{DR} \Delta q(X, \upsilon) \right) 1(\Delta S' \alpha \geq 0) \right] \Big|_{\alpha = a(\upsilon)} \Bigg)' \phi_i(\upsilon) d\upsilon, \\
R_{2i}^{+} = & \mathcal{D}^{+'} G^{-1} \psi^J(X_i, T_i, Z_i) e_i, \text{ with } G \equiv \mathbb{E}\left[\psi^J(X, T, Z) \psi^J(X, T, Z)' \right] = \mathbb{E}\left[\Psi' \Psi / n \right], \\
& \mathcal{D}^+ = \mathcal{D}_1^+ - \mathcal{D}_0^+, \mathcal{D}_z^+ = \int_0^1 \mathbb{E}\left[\psi^J(X, q_z(X, \upsilon), z) \chi^+(X, \upsilon) \right] d\upsilon, \\
R_{3i}^{+} = & \int_0^1 \left(\Delta m(X_i, \upsilon) - \pi_+^{DR} \Delta q(X_i, \upsilon) \right) \chi^+(X_i, \upsilon) d\upsilon, \\
B_+ = & \int_0^1 \int_{\mathcal{X}} \Delta q(x, \upsilon) \chi^+(x, \upsilon) f(x) dx d\upsilon.
\end{aligned}
$$

Let $\chi^-(x, \upsilon) = 1(\Delta q(x, \upsilon) < 0)$ and $B_- = \int_0^1 \int_{\mathcal{X}} \Delta q(x, \upsilon) \chi^-(x, \upsilon) f(x) dx d\upsilon$. Let $B = B_+ - B_-$.

For π_-^{DR}, define R_i^- as R_i^+ by replacing $+$ with $-$ in all the components in R_i^+.

For π^{DR}, define $R_i = R_{1i} + R_{2i} + R_{3i}$, where $R_{ki} = R_{ki}^+ - R_{ki}^-$ for $k = 1, 2, 3$ except that one needs to replace π_+^{DR} and π_-^{DR} with π^{DR} in R_{ki}^+ and R_{ki}^-, $k = 1, 3$.

S.7.2 Implementation

We estimate σ^2 by the sample analogue plug-in estimator, i.e., $\hat{\sigma}^2 = \hat{\sigma}_1^2 + \hat{\sigma}_2^2 + \hat{\sigma}_3^2$, where $\hat{\sigma}_k^2 = n^{-1} \sum_{i=1}^n \hat{R}_{ki}^2 / \hat{B}^2$, $\hat{B}$ and $\hat{R}_{ki}$ are consistent estimators for B and R_{ki} for $k = 1, 2, 3$, respectively, given in (S.10):

For $\hat{R}_{1i}$, $\partial_t \hat{m}_z$ is directly computed from Step 2. From the linear quantile regression literature, it is standard $\hat{\vartheta}(\upsilon) = n^{-1} \sum_{i=1}^n \hat{f}_{T|X,Z}(S_i' \hat{a}(\upsilon) | X_i, Z_i) S_i S_i'$. The derivative $\frac{\partial}{\partial \alpha} \mathbb{E}\big[\Delta m(X, \upsilon) 1(\Delta S_i' \alpha \geq 0) \big] \big|_{\alpha = a(\upsilon)}$ may be estimated by a numerical differentiation, i.e., $n^{-1} \sum_{i=1}^n \Delta \hat{m}(X_i, \upsilon) \big(1(\Delta S_i' (\hat{a}(\upsilon) + \imath/2) \geq 0) - 1(\Delta S_i' (\hat{a}(\upsilon) - \imath/2) \geq 0) \big) / \imath$ for some small $\imath > 0$.

For $\hat{R}_{2i}$, let $\hat{e}_i = Y_i - \psi^J(X_i, T_i, Z_i)' \hat{c}$, $\hat{\Omega} = n^{-1} \sum_{i=1}^n \hat{e}_i^2 \psi^J(X_i, T_i, Z_i) \psi^J(X_i, T_i, Z_i)'$, $\hat{G} = \Psi' \Psi / n$, and $\hat{\mho} = \hat{G}^{-1} \hat{\Omega} \hat{G}^{-1}$. Let $\hat{\mathcal{D}}^+ = n^{-1} \sum_{i=1}^n l^{-1} \sum_{\upsilon \in V^{(l)}} \hat{\psi}_i^J \hat{\chi}^+(X_i, \upsilon)$. Let $\hat{\mathcal{D}} = \hat{\mathcal{D}}^+ - \hat{\mathcal{D}}^-$. Then $\hat{\sigma}_{2n}^2 = \hat{\mathcal{D}}' \hat{\mho} \hat{\mathcal{D}}$, $\hat{\sigma}_{+2}^2 = \hat{\mathcal{D}^+}' \hat{\mho} \hat{\mathcal{D}^+}$, and $\hat{\sigma}_{-2}^2 = \hat{\mathcal{D}^-}' \hat{\mho} \hat{\mathcal{D}^-}$.

$\hat{R}_{3i}^{+} = l^{-1} \sum_{\upsilon \in V^{(l)}} \left(\Delta \hat{m}(X_i, \upsilon) - \hat{\pi}_{+}^{DR} \Delta \hat{q}(X_i, \upsilon) \right) \hat{\chi}^{+}(X_i, \upsilon)$, and $\hat{B}_{+}$ is analogous.

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